# SPECTRAL STRUCTURE OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS WITH CONSTANT ENERGY AND ENSTROPHY

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Abstract. Motivated by open questions in the long-time dynamics of fluid motion and fluid turbulence, we investigate the existence of nonstationary solutions to the Navier-Stokes equations (NSE) with constant energy profiles, with constant enstrophy profiles, or both. In this paper we make progress in this area by constructing a nonstationary solution to the incompressible NSE (on the 3D torus) whose energy remains constant. Similarly, we construct a nonstationary solution whose enstrophy remains constant. These constructions necessarily exist outside of the attractor and are supported on an infinite number of modes. On the 2D torus we show that when the force is an eigenvector of the Stokes operator any solution with nontrivial nonlinear term must be supported on an infinite number of Fourier modes. This result is then used to disprove the existence of the so-called *chained ghost solutions* introduced in [24].

## 1. Introduction

Our interest in the issue of the existence non steady-state solutions possessing time-independent global quantities is motivated by the need to understand the long-time dynamics of the equations of fluid dynamics in the context of fluid turbulence. In particular, it becomes important to describe the dynamics in terms of the key physical quantities involved in the empirical theory of turbulence dating back to Kolmogorov in 3D [17, 18] and Kraichnan in 2D [19] (see also [15] for a comprehensive overview and [7, 12] for a mathematical setup in terms of Navier-Stokes equations). Among these quantities, the key role is played by fluid energy, enstrophy, and in 2D, palinstrophy, which in the empirical theory determine behavior of such quantities as the Reynolds number, the energy dissipation rate, and the structure of energy transfer among the scales of the fluid flow. Another key feature of turbulence theories is the basic finite-dimensionality of fluid turbulence. This finite-dimensionality can be understood through the notion of Landau-Lifschitz degrees of freedom [20] and determining modes [2] – the maximal number of parameters needed to completely resolve dynamics of the fluid flow, which in turbulence is related to dissipation scale. To connect these empirical theories to the basic equations of motion, it becomes crucial to Investigate the long-time dynamics of the energy, enstrophy, and palinstrophy of the solutions of Navier-Stokes equations (NSE).

A natural focus of the mathematical studies of turbulence has been the global attractor of the 2D Navier-Stokes system (or weak attractor in 3D), which captures the long-term dynamics. In particular, the finite-dimensionality of the global attractor in 2D has been established [3, 21], and the above-mentioned physical quantities of energy, enstrophy, and palinstrophy of solutions on the attractor in the 2D periodic case under a stationary body force (and how they relate to turbulence) have been studied extensively ([1, 4, 5, 6, 8, 10, 11]). It was shown, for example, in [6] that the global attractor for the 2D NSE with periodic boundary conditions is bounded in the (suitably normalized) energy and enstrophy plane between a parabola and a line and that these bounds on the attractor location are sharp. Further refinements on these bounds and results in terms of the palinstrophy can be found in ([4, 5]).

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A question that arises naturally in this area is how much we can determine regarding the dynamics of the system by simply considering the energy, enstrophy, and palinstrophy of solutions in the attractor. If there can exist nonstationary solutions in the attractor where the energy, enstrophy, and palinstrophy nevertheless remain constant, then it would seem that these quantities somehow do not capture the dynamics of the solutions as described in empirical theory of turbulence. If it can be shown, however, that any solution with constant energy, enstrophy, and palinstrophy is indeed a stationary solution, then these quantities certainly capture at least some relevant features of the dynamics of the system. While fully resolving the dynamics necessitates a large number of degrees of freedom and determining modes [2, 9, 11, 14, 16], there is an indication that, consistent with the empirical theories, the dynamics of energy, enstrophy, and palinstrophy may still tell us quite a bit.

The question of whether a nonstationary solution in the attractor may have constant energy and enstrophy for all time was introduced in [10]. In that paper the authors considered the NSE in 2D with periodic boundary conditions under the assumption that the force is an eigenvector of the Stokes operator. They referred to nonstationary solutions in the attractor with constant energy and enstrophy as "ghost solutions", alluding to the fact that their existence in general remained unknown. It is known that under certain conditions ghost solutions in the attractor are not possible. For example, if the Grashof number is small enough then the attractor consists of a single point ([6]). Similarly, if the force is an eigenvector of the Stokes operator associated with the smallest eigenvalue, then the attractor consists of a single point and no ghost solutions are possible. Investigation into the existence of ghost solutions has continued more recently in ([23], [24]), where a sub-class of ghost solutions with an additional stability property was introduced. Such solutions were dubbed *chained ghost solutions* (see Section 3 for the relevant definitions). In general, results as they pertain to ghost solutions are limited in scope. Indeed, it is not known in general whether there exist nonstationary solutions to the NSE under the more relaxed condition of simply having constant energy (or simply having constant enstrophy, or constant palinstrophy).

Curiously, our investigations into the existence of ghost solutions naturally connect to the issues of finite-dimensionality of the NSE flows. Typically, the finite-dimensionality is described in terms of fractal dimension, in terms of Landau-Lifschitz degrees of freedom, or in terms of determining modes. However, in the context of ghost solutions, we consider whether the finite-dimensionality can be fully manifested in terms of Fourier scales. Namely, one would want to investigate the possibility that NSE solutions evolve in a finite-dimensional subspace of Fourier space – the so called finite-mode solutions. So far, the only such solutions are proved to be stationary solutions arising in special cases, e.g. in the case of single-mode force, which exclude turbulence ([6, 10, 22]). The question of the existence of non-stationary finite-mode solutions remains largely unresolved, yet resolving this question would not only improve our understanding of the finite-dimensionality of fluid flows, but also would show whether some NSE flows are described by an explicit finite-dimensional ODE in Fourier space.

In this paper we investigate the properties of nonstationary solutions to the Navier-Stokes equations with constant energy profiles and with constant enstrophy profiles, with a particular focus on a possibility of finite-mode solutions. Consistent with previous studies in this area ([6, 10, 23, 24]) we focus on the space periodic case with a constant-in-time force. First, we analyze such solutions under the simpler Stokes system and obtain an explicit description of such solutions. We then export results from that analysis to the NSE on the 3D torus by exploiting basic cancellations in the non-linearity, thus establishing the existence of 3D NSE solutions whose energy (respectively, enstrophy) remains constant. These constructions necessarily exist outside of the 3D (weak) attractor and are supported on an infinite number of Fourier modes. We then investigate the possibility of finite-mode solutions to the NSE on the 2D torus. As a result of this investigation we show that any solution to the NSE on the 2D torus with nontrivial nonlinear term must, if it exists, be supported on an infinite number of modes when the force is an eigenvector of the Stokes operator (we note that in this case, if the nonlinearity is zero, the structure of the finite-mode solutions is trivial). Together these results rule out the existence of so-called chained ghost solutions (as defined in [23]) since such solutions are necessarily finite-mode and cannot exist if the nonlinear term is zero.

In Section 2 we outline the theoretical framework for our discussion. In Section 3 we provide an overview of recent results in the literature. In Section 4 we analyze the Stokes system and use these results to construct a nonstationary constant-energy (as well as constant-enstrophy) solution to the 3D Navier-Stokes equations with periodic boundary conditions. In Section 5 we prove results regarding the possible forces that allow for finite-mode solutions to the 2D Navier-Stokes equations with periodic boundary conditions. We then apply these results to show that if the force is an eigenvector of the Stokes operator then solutions with nontrivial nonlinear term, if they exist, are necessarily supported on an infinite number of modes.

## 2. Preliminaries

In this paper we focus on the incompressible Navier-Stokes equations (NSE) with zero space average

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F \\ \nabla\cdot u = 0 \\ \int_\Omega u dx = 0, \quad \int_\Omega F dx = 0 \\ u(x,0) = u_0 \end{cases} \tag{1}$$

on a periodic spatial domain $\Omega = [0,L]^n_{per}$, $n = 2,3$. Here $u = \mathbf{u}(\mathbf{x},t)$ is the fluid velocity vector field, $\nu$ is the kinematic viscosity, $p = p(\mathbf{x},t)$ is the pressure (per density), $F = \mathbf{F}(\mathbf{x})$ is the body force vector field (per density), and $u_0 = \mathbf{u}_0(\mathbf{x})$ is the initial condition. The unknowns in these equations are the vector $u$ and the scalar $p$. In the space-periodic case the zero space average assumption is made without loss of generality (see, for example, [12], Chapter 2, Section 2).

Two important physical quantities that we will deal with are the kinetic energy and the enstrophy (per unit mass) of the fluid:

$$\text{energy}: \quad \frac{1}{2}\int_\Omega |u(x)|^2 dx \tag{2}$$

$$\text{enstrophy}: \quad \sum_{i=1}^n \frac{1}{2}\int_\Omega |\nabla u_i(x)|^2 dx. \tag{3}$$

We associate with this system the natural Hilbert space of divergence-free zero space average functions with bounded energy, often referred to as $H$. The inner product on $H$ is the usual $[L^2(\Omega)]^n$ inner product defined by

$$(u,v) = \int_\Omega u(x)\cdot v(x) dx. \tag{4}$$

For the norm on $H$ we use the notation

$$|u| = (u,u)^{1/2} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2} \tag{5}$$

where the difference between whether $|\cdot|$ refers to the norm in $H$ or the modulus of a vector is hopefully clear from context.

We view the system in (1) as an evolution equation in $H$. Taking the Helmoltz-Leray projection onto $H$ of (1) we have the following functional formulation of the Navier-Stokes equations:

$$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = f \\ u(0) = u_0 \end{cases} \tag{6}$$

where $A := \mathcal{P}_L(-\Delta u)$ is the Stokes operator, $B(u,v) := \mathcal{P}_L((u\cdot\nabla)v)$ is a bilinear operator, and $f := \mathcal{P}_L(F)$ (here $\mathcal{P}_L$ represent the Helmholtz-Leray projection). Note that $\mathcal{P}_L(\nabla p) = 0$.

The Stokes operator $A : D(A) \mapsto H$ is well-known to be self-adjoint with compact inverse, and with eigenvalues of the form

$$\lambda_j = \left(\frac{2\pi}{L}\right)^2 j\cdot j; \qquad j \in \mathbb{Z}^n\backslash\{0\}, \tag{7}$$

in the space-periodic case. The smallest eigenvalue of $A$ will play a special role in our estimates, so we will denote

$$\lambda_0 = \left(\frac{2\pi}{L}\right)^2. \tag{8}$$

We may write any vector field $u(x) \in H$ as follows:

$$u(x) = \sum_{j\in\mathbb{Z}^n\backslash\{0\}} u_j \omega_j(x) \tag{9}$$

with $u_j \in \mathbb{R}$ and where $\{\omega_j\}$ forms an orthonormal basis for $H$, with each $\omega_j$ being an eigenvector of $A$ with explicit eigenvalue $\lambda_j$. Indeed, in the space-periodic case the eigenvectors of $A$ may be represented by trigonometric polynomials. Thus, any function in $H$ may alternatively be written as a Fourier series

$$u = \sum_{j\in\mathbb{Z}^n\backslash\{0\}} \hat{u}_j e^{i(2\pi/L)j\cdot x} \tag{10}$$

where $\hat{u}_j \in \mathbb{C}^n$ such that $\sum_{j\in\mathbb{Z}^n\backslash\{0\}} |\hat{u}_j|^2 < \infty$ (to ensure finite energy), $\hat{u}_j = \overline{\hat{u}}_{-j}$ (to assure $u$ is real), and $\hat{u}_j \cdot j = 0$ (to assure that $u$ is divergence-free) for all $j \in \mathbb{Z}^n\backslash\{0\}$. Note that we have

$$|u|^2 = \sum_{j\in\mathbb{Z}^n\backslash\{0\}} |\hat{u}_j|^2. \tag{11}$$

Another natural space that we associate with this system is the subspace of $H$ consisting of finite-enstrophy functions, often referred to as $D(A^{1/2})$ or $V$. We write the natural norm on $V$, which is equivalent to the $[H^1(\Omega)]^n$ norm, as

$$\|u\| = (A^{1/2}u, A^{1/2}u)^{1/2} = ((u,u))^{1/2} = \left(\int_\Omega \left|\sum_{i=1}^n \frac{\partial}{\partial x_i} u(x)\right|^2\right)^{1/2}. \tag{12}$$

Note that we have

$$\|u\|^2 = \sum_{j\in\mathbb{Z}^n\backslash\{0\}} |j|^2 |\hat{u}_j|^2. \tag{13}$$

For strong solutions to the NSE we have the following energy balance equation:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 = (f,u). \tag{14}$$

This is obtained by taking the inner product in $H$ of the the Navier-Stokes equations with the solution $u$ and using the fact that $(B(u,u),u) = 0$. In two dimensions only, we also have the following enstrophy balance equation

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 = (f, Au), \tag{15}$$

which is obtained by taking the inner product of the NSE with $Au$ and using the two-dimensional orthogonality relation $(B(u,u), Au) = 0$.

We may represent the system in (6) as an infinite-dimensional system of coupled ordinary differential equations by rewriting it in terms of its Fourier series:

$$\hat{u}_j'(t) + \nu\lambda_j\hat{u}_j(t) + \widehat{B(u,u)}_j(t) = \hat{f}_j; \qquad j \in \mathbb{Z}^n\backslash\{0\}. \tag{16}$$

Given $u, v \in D(A)$ we may explicitly write the Fourier coefficients of $B(u,v) \in H$ as follows:

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^n\backslash\{0\}}\left[(\hat{u}_{j-k}\cdot k)\hat{v}_k - \frac{(\hat{u}_{j-k}\cdot k)(\hat{v}_k\cdot j)}{|j|^2}j\right]. \tag{17}$$

In 2D we may alternatively write the Fourier coefficients of $B(u,v)$ as follows:

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^2\backslash\{0\}}\frac{(\hat{u}_{j-k}\cdot k)(\hat{v}_k\cdot j^\perp)}{|j|^2}j^\perp \tag{18}$$

where for $j = (j_1, j_2)$ we define $j^\perp = (-j_2, j_1)$. See the Appendix (Section 6) for formal calculations of (17) and (18).

Two dimensionless quantities that are useful for understanding fluid flow are the generalized Grashof number (introduced in [13]) and the well-known Reynolds number. The definition of the generalized Grashof number depends on the number of spatial dimensions of the flow. It is defined as

$$G = \frac{|f|}{\nu^2\lambda_0} \text{ in 2D}; \qquad G = \frac{|f|}{\nu^2\lambda_0^{3/4}} \text{ in 3D}. \tag{19}$$

In order to define the Reynolds number one must first define some appropriate (time-independent) "average" of the fluid velocity. Several reasonable choices are possible, so let $\langle|u|\rangle$ refer to an appropriate average of the fluid speed. Then we define the Reynolds number as

$$Re = \frac{\langle|u|\rangle}{\nu\lambda_0^{1/2}}. \tag{20}$$

In two-dimensions there exists a well-defined global attractor (a compact subset of $H$ that is invariant under the flow and that uniformly attracts bounded sets in $H$). In 2D, we may define the solution operator $S(t)$ by $S(t)u_0 = u(t)$, where $u(t)$ is the unique solution to (6) at time $t$ with initial data $u_0$. It is well-known that the operator $S(t)$ depends continuously on the initial data.

## 3. Specific Preliminaries

The term *ghost solution* was first introduced in [10] to refer to nonstationary solutions to the NSE whose energy and enstrophy remain constant for all time. In the original context the spatial domain of the fluid flow is the 2D torus, the force is an eigenvector of the Stokes operator, and the solution lies in the global attractor. However, the concept of a ghost solution makes sense even outside of these specifications. Thus, we define ghost solutions as follows:

**Definition 3.1.** *A* **ghost solution** *is a nonstationary solution, $u(x,t)$, to (6) (in 2D or 3D) such that $|u(t)|$ and $\|u(t)\|$ are constant in time for all $t \geqslant 0$ where $u$ is defined.*

Note that this definition allows the possibility of ghost solutions that exist outside of the attractor. The only restriction on the forcing function $f$ is that it is a stationary vector in $H$. While in this paper we work on a periodic domain, that is not an essential feature of the definition.

The advantage of working on a 2D periodic domain with an eigenvector force is that in this case ghost solutions have the following important property:

**Theorem 3.2** (See equation (6.3) of [10])**.** *Let $\Omega = [0, L]^2_{per}$ and let $Af = \lambda_f f$. Let $u$ be a ghost solution. Then the following relationship holds:*

$$\nu\|u\|^2 = (f, u) = \frac{\nu}{\lambda_f}|Au|^2. \tag{21}$$

This relationship is an immediate consequence of the energy and enstrophy balance equations in 2D. It also directly shows that, under these conditions, constant energy and constant enstrophy together imply constant palinstrophy ($|Au|^2$). As a result, the authors of [24] investigate the following (dynamic in time) subspace of $H$

$$\tilde{H}(t) := \operatorname{span}\{f, u(t), Au(t), A^2u(t)\} \tag{22}$$

since for any vector $v \in \tilde{H}$ the product $(v, u)$ is constant. The authors of [24] show that for any ghost solution $u$ it must be the case that $\operatorname{span}\{f\} \subsetneq \operatorname{span}\{f, u\} \subsetneq \operatorname{span}\{f, u, Au\}$, but their work allows for the possibility that $A^2u \in \operatorname{span}\{f, u, Au\}$. Considering this potential degeneracy motivates the following definition.

**Definition 3.3** (See [24] Definition 6.1)**.** *Consider the system (6) with $\Omega = [0, L]^2_{per}$ and suppose that $f$ is an eigenvector of the Stokes operator. A* **chained ghost solution** *is a ghost solution in the global attractor satisfying the following relation:*

$$A^2u(t) = \gamma f + \beta u(t) + \alpha Au(t), \quad \forall t \in \mathbb{R},$$

*for real scalars $\alpha, \beta$, and $\gamma$.*

Unlike the general notion of ghost solutions, the motivation for considering chained ghost solutions necessarily relies on the domain being the 2D torus and the force being an eigenvector of the Stokes operator.

The authors of [24] prove several results concerning chained ghost solutions. The most relevant result for this paper is the fact that chained ghost solutions may be decomposed as a sum of three eigenvectors of the Stokes operator. This theorem was restated nicely in [23]. We state a simplified version of the result here:

**Theorem 3.4** (See Theorem 6.3 in [24] or Theorem 2.1 in [23])**.** *A chained ghost solution $u(t)$ may be written in the following form:*

$$u(t) = u_+(t) + u_-(t) + \frac{\|u\|^2}{|f|^2}f,$$

*where $u_+$ and $u_-$ are eigenvectors of the Stokes operator $A$.*

Since $\frac{\|u\|^2}{|f|^2}$ is a scalar and $f$ is assumed to be an eigenvector of the Stokes operator, this theorem implies that any chained ghost solution may be written as a sum of three eigenvectors of $A$.

## 4. Nonstationary Constant-Energy Solution Construction on the 3D Torus

In this section we construct a nonstationary solution to (6) in 3D that has constant energy everywhere it is defined. We begin by constructing a nonstationary solution to the Stokes system (*i.e.*, the Navier-Stokes equations but without the nonlinear term) that is defined and has constant energy for all $t \geqslant 0$. We then show that this construction can be modified to create a nonstationary constant-energy solution the full Navier-Stokes equations in 3D.

4.1. **Stokes System.** By the *Stokes system* we refer to the following set of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = F \\ \nabla \cdot u = 0 \\ \int_\Omega u dx = 0, \quad \int_\Omega F dx = 0 \\ u(x,0) = u_0 \end{cases} \tag{23}$$

on a periodic spatial domain $\Omega = [0,L]^n_{per}$, $n = 2,3$. This system is simply the NSE without the non-linear term. The functional formulation of the Stokes system in $H$ is as follows:

$$\begin{cases} \frac{du}{dt} + \nu A u = f \\ u(0) = u_0. \end{cases} \tag{24}$$

Recall that we may write any function in $H$ as follows:

$$u(x) = \sum_{j \in \mathbb{Z}^n \backslash \{0\}} u_j \omega_j(x) \tag{25}$$

where $\{\omega_j\}_{j\in\mathbb{Z}^2\backslash\{0\}}$ is a set of orthonormal eigenvectors of $A$ spanning $H$ with explicit eigenvalues $\lambda_j = \left(\frac{2\pi}{L}\right)^2 |j|^2$. Considering this eigenvector expansion, we see that (24) is equivalent to the following (possibly infinite) system of linear ordinary differential equations:

$$\frac{d}{dt} u_j(t) + \nu\lambda_j u_j(t) = f_j, \qquad j \in \mathbb{Z}^n \backslash \{0\}. \tag{26}$$

Each of these equations can be solved explicitly for all $t \geqslant 0$:

$$u_j(t) = \left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right) e^{-\nu\lambda_j t} + \frac{f_j}{\nu\lambda_j}. \tag{27}$$

From here it can be seen that the global attractor for this system consists of the steady-state solution: $u^* = \frac{1}{\nu} A^{-1} f$ (*i.e.* $u_j^* = \frac{f_j}{\nu\lambda_j}$ for each $j \in \mathbb{Z}^2\backslash\{0\}$) since solutions in the attractor must be bounded for all time. Note that the fact that the attractor is a single point clearly implies non-existence of ghost located solutions on the attractor. However, we will consider the question in a more general setting of Definition 3.1.

In what follows we will construct a nonstationary constant-energy solution to the Stokes system. For specificity, we work on the 2D torus with $\Omega = [0,L]^2_{per}$, but this construction can be trivially adapted to $\mathbb{R}^n$, $n \geqslant 2$. We begin with a lemma regarding a necessary and sufficient condition for the existence of nonstationary constant-energy solutions to the Stokes system.

**Lemma 4.1.** *Any nonstationary constant-energy solution to the Stokes system (24) requires that the force and initial condition satisfy the following condition:*

$$\sum_{j\in\mathbb{Z}^2\backslash\{0\}} \left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t} + \frac{2f_j}{\nu\lambda_j}\left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right) e^{-\nu\lambda_j t} = 0 \quad \text{for all } t \geqslant 0, \tag{28}$$

*where at least one $u_j$ is such that $u_j(t) \neq \frac{f_j}{\nu\lambda_j}$. The converse also holds.*

*Proof.* The energy of the solution to (24) is given by

$$\begin{aligned}\frac{1}{2}|u(t)|^2 &= \frac{1}{2}\sum_{j\in\mathbb{Z}^2\setminus\{0\}} u_j(t)^2\\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}^2\setminus\{0\}}\left[\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)e^{-\nu\lambda_j t}+\frac{f_j}{\nu\lambda_j}\right]^2\\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}^2\setminus\{0\}}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t}+\frac{2f_j}{\nu\lambda_j}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)e^{-\nu\lambda_j t}+\left(\frac{f_j}{\nu\lambda_j}\right)^2.\end{aligned}$$

If the energy of a solution is to remain constant then this solution must have the same energy as the solution in the global attractor. That is, a constant-energy solution must have energy equal to $\frac{1}{2}\sum_{j\in\mathbb{Z}^2}\left(\frac{f_j}{\nu\lambda_j}\right)^2$. Such a solution would then require that

$$\sum_{j\in\mathbb{Z}^2\setminus\{0\}}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t}+\frac{2f_j}{\nu\lambda_j}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)e^{-\nu\lambda_j t}=0. \tag{29}$$

Note that if we try to make the sum zero by making all the terms zero, this requires that we set $u_j(0)=\frac{f_j}{\nu\lambda_j}$ for each $j$. However, if this is the case then we have that each $u_j(t)$ is constant in time, and in fact that $u(t)$ is the steady-state solution. Thus, in order for $u(t)$ to be nonstationary, we need for $u_j(t)\neq\frac{f_j}{\nu\lambda_j}$ for at least one $j$.

For the reverse implication note that if $\sum_{j\in\mathbb{Z}^2\setminus\{0\}}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t}+\frac{2f_j}{\nu\lambda_j}\left(u_j(0)-\frac{f_j}{\nu\lambda_j}\right)e^{-\nu\lambda_j t}=0$ then $\frac{1}{2}|u(t)|^2=\frac{1}{2}\sum_{j\in\mathbb{Z}^2}\left(\frac{f_j}{\nu\lambda_j}\right)^2$ and the energy of $u$ is constant. Again, the condition that some $u_j$ is such that $u_j(t)\neq\frac{f_j}{\nu\lambda_j}$ guarantees that $u$ is nonstationary. □

Suppose $u$ is a nonstationary constant-energy solution to (24). Let $j_0$ be such that $u_{j_0}(0)\neq\frac{f_{j_0}}{\nu\lambda_{j_0}}$. Then we at least have two terms in equation (28) that need to be cancelled:

$$\left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t}; \qquad \& \qquad \frac{2f_{j_0}}{\nu\lambda_{j_0}}\left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)e^{-\nu\lambda_{j_0}t}. \tag{30}$$

Since the functions $e^{-2\nu\lambda_{j_0}t}$ and $e^{-\nu\lambda_{j_0}t}$ are linearly independent, we know that

$$\left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t}+\frac{f_{j_0}}{\nu\lambda_{j_0}}\left(u_{j_0}(0)-\frac{2f_{j_0}}{\nu\lambda_{j_0}}\right)e^{-\nu\lambda_{j_0}t}\not\equiv 0. \tag{31}$$

Thus, we conclude that the term $\left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t}$ must be cancelled by terms of the form $\frac{2f_k}{\nu\lambda_k}\left(u_k(0)-\frac{f_k}{\nu\lambda_k}\right)e^{-\nu\lambda_k t}$ where $\lambda_k=2\lambda_{j_0}$. However, this creates new terms $\left(u_k(0)-\frac{f_k}{\nu\lambda_k}\right)^2 e^{-2\nu\lambda_k t}$, which themselves must be cancelled by terms associated with an eigenvalue equal to $2\lambda_k=4\lambda_{j_0}$. These considerations motivate the following lemma.

**Lemma 4.2.** *For any $j\in\mathbb{Z}^2\setminus\{0\}$ and $n\in\mathbb{N}$ we have that $2^n\lambda_j$ is an eigenvalue of the Stokes operator.*

*Proof.* This follows almost directly from the Sum of Two Squares Theorem, which states that an integer $m$ is the sum of two squares if and only if, in the prime factorization of $m$, all primes congruent to 3 modulo 4 are to an even power. Consider an arbitrary eigenvalue $\lambda_j = \left(\frac{2\pi}{L}\right)^2 j \cdot j$. Since $j \cdot j = j_1^2 + j_2^2$ is the sum of two squares, the prime factorization of $j \cdot j$ contains only even powers of primes congruent to 3 modulo 4. Note also that the prime factorization of $2^n j \cdot j$ then also has only even powers of primes congruent to 3 modulo 4 for any $n$. Thus $2^n j \cdot j = k_1^2 + k_2^2$ for some integers $k_1, k_2$. Thus, for $k = (k_1, k_2)$ we have $2^n \lambda_j = \lambda_k$. □

**Remark 4.3.** *In the 3D case, by Legendre's Theorem, $m = j \cdot j$ is a sum of three squares if and only if $m$ is not of the form $4^k(8i+7)$, and therefore the 3D version of Lemma 4.2 holds for any $j \in \mathbb{Z}^2 \backslash \{0\}$ such that $j \cdot j \not\equiv 7 (\bmod 8)$ (e.g. $j = (1,0,0)$). In the case $n \geqslant 4$, by Lagrange's theorem, any $m \in \mathbb{N}$ can be represented as a sum of four, and consequently of any number of squares, and therefore Lemma 4.2 holds with no restrictions on $j \in \mathbb{Z}^n \backslash \{0\}$.*

**Remark 4.4.** *For an explicit construction of a sequence of eigenvalues $\{2^n \lambda_j\}_{n \in \mathbb{N}}$, consider the following:*

*For $j = (j_1, 0)$ with $j_1 \neq 0$ let*

$$2^n \lambda_j = \begin{cases} \lambda_{(j_1 2^{(n-1)/2},\, j_1 2^{(n-1)/2})} & \text{if } n \text{ is odd} \\ \lambda_{(j_1 2^{n/2},\, 0)} & \text{if } n \text{ is even} \end{cases}$$

Before we move on to the next lemma, let us consider the following motivating calculations. Let $j_0$ be such that $f_{j_0} = 0$ and $u_{j_0}(0) \neq 0$. Then the $j_0^{\text{th}}$ term in the sum (28) would simply be

$$u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0} t}.$$

Let $\lambda_{j_1}$ be such that $\lambda_{j_1} = 2\lambda_{j_0}$. Then we have that the $j_1^{\text{st}}$ term in sum (28) is

$$\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1} t} + \frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) e^{-\nu\lambda_{j_1} t}.$$

Since $\lambda_{j_1} = 2\lambda_{j_0}$ we have that $\frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) e^{-\nu\lambda_{j_1} t}$ from the $j_1^{\text{st}}$ term cancels the $j_0^{\text{th}}$ term $u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0} t}$ exactly when

$$\frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) = -u_{j_0}(0)^2,$$

or rather

$$u_{j_1}(0) = \frac{f_{j_1}}{2\nu\lambda_{j_0}} - \frac{\nu\lambda_{j_0} u_{j_0}(0)^2}{f_{j_1}}.$$

Of course this leaves us with the first part of the $j_1^{\text{st}}$ term leftover: $\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1} t}$. If we let $\lambda_{j_2}$ be such that $\lambda_{j_2} = 2\lambda_{j_1} = 4\lambda_{j_0}$ then the $j_2^{\text{nd}}$ term of sum (28) is

$$\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right)^2 e^{-2\nu\lambda_{j_2} t} + \frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right) e^{-\nu\lambda_{j_2} t}.$$

This time we get that the second part of the $j_2^{\text{nd}}$ term, $\frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0)-\frac{f_{j_2}}{\nu\lambda_{j_2}}\right)e^{-\nu\lambda_{j_2}t}$, cancels the remainder from the $j_1^{\text{st}}$ term, $\left(u_{j_1}(0)-\frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1}t}$, exactly when

$$\frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0)-\frac{f_{j_2}}{\nu\lambda_{j_2}}\right) = -\left(u_{j_1}(0)-\frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 = -\frac{\nu^2\lambda_{j_0}^2 u_{j_0}(0)^4}{f_{j_1}^2},$$

or rather

$$u_{j_2}(0) = \frac{f_{j_2}}{4\nu\lambda_{j_0}} - \frac{2\nu^3\lambda_{j_0}^3 u_{j_0}(0)^4}{f_{j_1}^2 f_{j_2}}.$$

Continuing in this pattern, we get successive cancellation of terms in the $j_n$ modes exactly when we define

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^n (f_{j_k})^{2^{n-k}}}.$$

We now establish the following partial sum lemma:

**Lemma 4.5.** *Fix $j_0 \in \mathbb{Z}^2\backslash\{0\}$. Choose a sequence $j_n \in \mathbb{Z}^2$, $n = 1, 2, 3\ldots$, such that $\lambda_{j_n} = 2^n\lambda_{j_0}$. Let $f_{j_0} = 0$ and let $f_{j_n} \neq 0$ for $n > 0$. Let $u_{j_0}(0) \neq 0$. If we define $u_{j_n}(0)$ for $n \geqslant 1$ recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^n (f_{j_k})^{2^{n-k}}}, \tag{32}$$

*then the formal sum*

$$\sum_{n=0}^{\infty}\left(u_{j_n}(0)-\frac{f_{j_n}}{\nu\lambda_{j_n}}\right)^2 e^{-2\nu\lambda_{j_n}t} + \frac{2f_{j_n}}{\nu\lambda_{j_n}}\left(u_{j_n}(0)-\frac{f_{j_n}}{\nu\lambda_{j_n}}\right)e^{-\nu\lambda_{j_n}t} \tag{33}$$

*has as its $N^{th}$ partial sum*

$$S_N = \left[\frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^N (f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t}. \tag{34}$$

*Proof.* We prove this by induction. As our base case we have

$$S_0 = \left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t} + \frac{2f_{j_0}}{\nu\lambda_{j_0}}\left(u_{j_0}(0)-\frac{f_{j_0}}{\nu\lambda_{j_0}}\right)e^{-\nu\lambda_{j_0}t} = u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0}t},$$

which is of the correct form. Now suppose we have that

$$S_N = \left[\frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^N (f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t}$$

for $N \geqslant 0$. Distributing the square allows us to write this in a more convenient form for later:

$$S_N = \frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^{N+1}}}{2^{2(N+1)}(\nu\lambda_{j_0})^2\prod_{k=1}^N (f_{j_k})^{2^{N+1-k}}}e^{-2^{N+1}\nu\lambda_{j_0}t} \tag{35}$$

Consider the following calculations:

$$\begin{aligned}
S_{N+1} &= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} + \frac{2f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} - \frac{2f_{j_{N+1}}}{\nu 2^{N+1}\lambda_{j_0}}\left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{N+2}\nu\lambda_{j_0}\prod_{k=1}^{N+1}(f_{j_k})^{2^{N+1-k}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} - \left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{2N+2}(\nu\lambda_{j_0})^2\prod_{k=1}^{N}(f_{j_k})^{2^{N+1-k}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} \\
&= \left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{(N+1)+1}\nu\lambda_{j_0}\prod_{k=1}^{N+1}(f_{j_k})^{2^{N+1-k}}}\right]^2 e^{-2^{(N+1)+1}\nu\lambda_{j_0}t}.
\end{aligned}$$

In the move to the second line we use the definition of $u_{j_{N+1}}$ as given in equation (32). The move to the third line is the result of simplification after using the fact that $\lambda_{j_{N+1}} = 2^{N+1}\lambda_{j_0}$. The move to the fourth line uses the formulation of $S_N$ given in equation (35). The final line again uses the definition of $u_{j_{N+1}}$ as given in equation (32). Thus we have that $S_{N+1}$ is of the correct form and the lemma is established. □

**Remark 4.6.** *Choose $(j_n)$, $f_{j_n}$, and $u_{j_n}(0)$ as in Lemma 4.5. If we additionally choose $f_j = 0$ and $u_j(0) = 0$ for all $j$ not in the sequence $\{j_n\}$, then we have the sum from (28) reduces to (33). For such choices of our parameters, we have that the identity (28) is satisfied exactly when*

$$\lim_{N\to\infty}\left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} = 0. \tag{36}$$

In one sense, it is easy to find appropriate values of $\lambda_{j_0}, \nu, u_j(0)$ and $\{f_{j_n}\}$ such that the limit in (36) is satisfied for all $t \geqslant 0$ (for example, set all parameters equal to 1). However, we should minimally require that our choices guarantee that the force and initial condition are at least in $H$ and ideally, due our interest in the constant enstrophy which we address later, in $V$. Our task now is to find values of $\lambda_{j_0}, \nu, u_{j_0}$ and $\{f_{j_n}\}$ such that the following criteria hold:

(1) $u(0) \in V$

(2) $f \in V$

(3) $\lim_{N\to\infty}\left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} = 0.$

As a final preliminary result, we require the closed form of an important sum that arises in our calculations:

**Lemma 4.7.** *For any $N \in \mathbb{N}$*

$$\sum_{k=1}^{N} k2^{N-k} = 2^{N+1} - N - 2. \tag{37}$$

*Proof.* This can be shown by a straighforward proof by induction. □

We now prove the following theorem which establishes the construction of a constant-energy solution to the Stokes system.

**Theorem 4.8.** *Fix $j_0 \in \mathbb{Z}^2\backslash\{0\}$. Choose a sequence $j_n \in \mathbb{Z}^2$, $n = 1, 2, 3\ldots$, such that $\lambda_{j_n} = 2^n\lambda_{j_0}$. Let $f_{j_n} = \frac{1}{b^n}$ for $n > 0$ with $b > \sqrt{2}$, and let $f_j = 0$ for all other $j \in \mathbb{Z}^2\backslash\{0\}$. Let $u_{j_0}(0) \neq 0$ be such that $2\nu\lambda_{j_0}|u_{j_0}(0)|\, b^2 \leqslant 1$, and define $u_{j_n}(0)$ recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n \lambda_{j_0}} - \frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}. \tag{38}$$

*Let $u_j(0) = 0$ for all other $j \in \mathbb{Z}^2\backslash\{0\}$. Then $f$ and $u(0)$ are in $V$ and define a force and initial condition such that the solution to the Stokes system (24) is nonstationary with constant energy for all $t \geqslant 0$.*

*Proof.* Let $f, u$ be defined as in the hypothesis of the theorem. First we show that $f \in V$. We calculate the norm of $f$ in $V$ as follows:

$$\begin{aligned}\|f\|^2 &= \sum_{j\in\mathbb{Z}^2\backslash\{0\}} \lambda_j f_j^2 \\ &= \sum_{n=1}^{\infty} \lambda_{j_n} f_{j_n}^2 \\ &= \sum_{n=1}^{\infty} 2^n\lambda_{j_0}\left(\frac{1}{b^n}\right)^2 \\ &= \lambda_{j_0}\sum_{n=1}^{\infty}\left(\frac{2}{b^2}\right)^n.\end{aligned}$$

Thus, we have $f \in V$ for $b > \sqrt{2}$.

Next we calculate a closed form for the term $\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}$. We have

$$\begin{aligned}\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}} &= \prod_{k=1}^{n}(b^k)^{2^{n-k}} \\ &= \prod_{k=1}^{n} b^{k2^{n-k}} \\ &= b^{\sum_{k=1}^{n} k2^{n-k}} \\ &= b^{2^{n+1}-n-2}\end{aligned}$$

where the last identity uses Lemma 4.7.

Now we calculate the norm of $u(0)$ in $V$:

$$\begin{aligned}
\|u(0)\|^2 &= \sum_{j\in\mathbb{Z}^2\setminus\{0\}} \lambda_j u_j(0)^2 \\
&= \sum_{n=1}^{\infty} \lambda_{j_n} u_{j_n}(0)^2 \\
&= \sum_{n=1}^{\infty} 2^n\lambda_{j_0}\left(\frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}\right)^2 \\
&= \sum_{n=1}^{\infty} 2^n\lambda_{j_0}\left(\frac{1}{\nu 2^n\lambda_{j_0}b^n} - \frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^n} b^{2^{n+1}-n-2}}{2^{n+1}\nu\lambda_{j_0}}\right)^2 \\
&= \sum_{n=1}^{\infty} 2^n\lambda_{j_0}\left(\frac{1}{\nu 2^n\lambda_{j_0}b^n} - \frac{(2\nu\lambda_{j_0}u_{j_0}(0)b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0}b^2b^n}\right)^2 \\
&= \sum_{n=1}^{\infty} 2^n\lambda_{j_0}\left(\left(\frac{1}{\nu 2^n\lambda_{j_0}b^n}\right)^2 - 2\left(\frac{1}{\nu 2^n\lambda_{j_0}b^n}\right)\left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0)b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0}b^2b^n}\right) + \left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0)b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0}b^2b^n}\right)^2\right).
\end{aligned}$$

In order for $\|u(0)\|^2$ to be finite we need this final sum to converge. Given the dominance of the terms with exponent $2^n$, we have convergence exactly when $2\nu\lambda_{j_0}|u_{j_0}(0)|\, b^2 \leqslant 1$. Thus, $u \in V$ provided $2\nu\lambda_{j_0}|u_{j_0}(0)|\, b^2 \leqslant 1$.

As mentioned in Remark 4.6, the necessary and sufficient condition for having a nonstationary constant-energy solution to the Stokes system is guaranteed in this case when the limit of the partial sums given by equation (34) in Lemma 4.5 is equal to 0.

Let us rewrite the $N^{\text{th}}$ partial sum given by equation (34) given our specific choice of $f$.

$$S_N = \left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} = \left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0)b^2)^{2^N}}{2^{N+1}\nu\lambda_{j_0}b^2b^N}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t}. \tag{39}$$

Notice that the requirement for $u \in V$, namely, $2\nu\lambda_{j_0}|u_{j_0}(0)|\, b^2 \leqslant 1$, is also sufficient to guarantee that $\lim_{N\to\infty} S_N = 0$ as desired.

□

**Remark 4.9.** *We may similarly construct a nonstationary solution to the Stokes system that has constant enstrophy. Define $f$ as in Theorem 4.8. Let $u_{j_0}(0) \neq 0$, and define $u_{j_n}(0)$ recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{(\nu\lambda_{j_0}u_{j_0}(0))^{2^n}}{2^n\nu\lambda_{j_0}\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}. \tag{40}$$

*Let $u_j(0) = 0$ for all other $j \in \mathbb{Z}^2\setminus\{0\}$. Then $f$ and $u(0)$ are in $V$ and define a force and initial condition such that the solution to the Stokes system (24) is nonstationary with constant enstrophy for all $t \geqslant 0$ provided $\nu\lambda_{j_0}|u_{j_0}(0)|\, b^2 \leqslant 1$. The proof mirrors the proof of Theorem 4.8.*

*Indeed, for any $s \in \mathbb{R}$ we may construct a solution to the Stokes system such that $|A^{s/2}u|$ is constant. For a given $s$ we define $u_{j_n}(0)$ recursively as follows:*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{(2^{1-s}\nu\lambda_{j_0}u_{j_0}(0))^{2^n}}{2^{n+1-s}\nu\lambda_{j_0}\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}. \tag{41}$$

*For $b^2 > 2^{(s/2)-1}$ we have $f \in D(A^{(s/2)-1})$ and $u(0) \in D(A^{s/2})$. These data define a force and initial condition such that the solution to the Stokes system (24) is nonstationary with constant enstrophy for all $t \geqslant 0$ provided that $2^{1-s}\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leqslant 1$.*

**Remark 4.10.** *In our construction of a nonstationary constant-energy solution to the Stokes system there exists a time $t < 0$ such that the energy is no longer well-defined. Note that the energy of our solution is finite so long as the limit of the partial sums given by equation (34) converges. The limit diverges for values of $t$ such that*

$$2\lambda_{j_0}\nu|u_{j_0}(0)|\, b^2 e^{-\nu\lambda_{j_0}t} > 1. \tag{42}$$

**Remark 4.11.** *In our definition of $u(0)$, the decay rate of the $f_{j_n}$ coefficients competes against the convergence of the sum defining $u(0)$. Choosing $f_{j_n}$ to be a geometric series caused the term $\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}$ to grow essentially at the rate of $(b^2)^{2^n}$. By chance, this matched the growth rate of the term $(\lambda_{j_0}\nu u_{j_0}(0))^{2^n}$ and allowed us to choose values of $\lambda_{j_0}, \nu, u_{j_0}(0)$, and $b$ such that convergence of the sum defining $u(0)$ is guaranteed. However, if the $f_{j_n}$ terms decay appreciably faster, then the sum defining $u(0)$ necessarily diverges. Thus, there is a limit to how smooth our choice of $f$ can be. For example, we may have $f \in D(A^s)$ for any $s \in \mathbb{R}$. However, $f$ cannot be in an "analytic" class such as $D(e^{A^s})$ for any $s \in \mathbb{R}$.*

We end this subsection with a pair of theorems regarding nonstationary constant-energy and nonstationary constant-enstrophy solutions to the Stokes system. Note that the following theorems are independent of the nonstationary-constant energy and nonstationary constant-enstrophy constructions provided above.

**Theorem 4.12.** *There does not exist a nonstationary solution to the Stokes system (24) with both constant energy and constant enstrophy, no matter what initial condition and time-independent force is chosen.*

*Proof.* Suppose $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$ for some $j_0 \in \mathbb{Z}^2\backslash\{0\}$ (this is the condition for $u$ to be nonstationary). Then in order to for $u$ to have constant energy we require that the sum in (28) be 0. Thus, we must have (at least) the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = -\sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right). \tag{43}$$

In order to have constant enstrophy we have the following requirement (analogous to Equation 28):

$$\sum_{j\in\mathbb{Z}^2\backslash\{0\}} \lambda_j \left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t} + \frac{2f_j}{\nu}\left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right)e^{-\nu\lambda_j t} = 0. \tag{44}$$

Thus, if $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$, then in order to have constant enstrophy we must have the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \lambda_{j_0}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = -\sum_{|\lambda_n|=2\lambda_{j_0}} 2\lambda_{j_0}\frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right),$$

or rather

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = 2\left(-\sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)\right). \tag{45}$$

Equations (43) and (45) can only be simultaneously satisfied if both sides of the equations are 0. □

**Theorem 4.13.** *If the Stokes system (24) admits a nonstationary constant-energy solution then that solution must be supported on an infinite number of eigenvectors of the Stokes operator. In addition, the force must also be supported on an infinite number of eigenvectors of the Stokes operator.*

*Proof.* Suppose for contradiction that $u$ is a nonstationary constant-energy solution to the Stokes system (24) that is supported on only a finite number of eigenvectors of the Stokes operator. As in Theorem 4.12, in order for $u$ to be nonstationary we require $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ for some $j \in \mathbb{Z}^2\backslash\{0\}$. Let $j_0$ be such that $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$ and for any other $j \in \mathbb{Z}^2\backslash\{0\}$ such that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ we have $|j| \leqslant |j_0|$. In order to for $u$ to have constant energy we require that the sum in (28) be 0, and so we must have (at least) the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = -\sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right). \tag{46}$$

Since the left-hand side is not zero, the right-hand side must also be non-zero. However, this means that there must exist a $j_1 \in \mathbb{Z}^2\backslash\{0\}$ such that $|j_1| > |j_0|$, $f_{j_1} \neq 0$, and $u_{j_1}(0) \neq \frac{f_j}{\nu\lambda_{j_1}}$. This contradicts our assumption on $j_0$. Thus $u$ must be supported on an infinite number of eigenvectors of the Stokes operator. Not only that, we must have that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ for an infinite number of $j$. As we just saw, for each $j$ such that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$, there must exist a $j_1$ such that $|j_1| > |j|$, $f_{j_1} \neq 0$, and $u_{j_1}(0) \neq \frac{f_j}{\nu\lambda_{j_1}}$. Thus $f$ must also be supported on an infinite number of eigenvectors of the Stokes operator. □

**Remark 4.14.** *The analogous result holds for constant-enstrophy solutions. That is, if the Stokes system (24) admits a nonstationary constant-enstrophy solution then the solution and the force must be supported on an infinite number of eigenvectors of the Stokes operator. Indeed, this holds for any nonstationary solution $u$ with $|A^{s/2}u|$ constant for some $s \in \mathbb{R}$.*

4.2. **Extension to 3D Navier-Stokes.** Recall that when working on the 3D torus, *i.e.* $\Omega = [0, L]^3_{per}$, we may write an element of $H$ in terms of its Fourier expansion as follows:

$$u(x) = \sum_{j\in\mathbb{Z}^3\backslash\{0\}} \hat{u}_j e^{i(2\pi/L)j\cdot x}$$

$\hat{u}_j \in \mathbb{C}^3$ such that $\overline{\hat{u}}_j = \hat{u}_{-j}$ and $j \cdot \hat{u}_j = 0$. As shown in Appendix 6, we may write the nonlinear term in (6) as

$$B(u, u) = \mathcal{P}_L[(u \cdot \nabla)u] = \mathcal{P}_L\left[\frac{i2\pi}{L} \sum_{j,k\in\mathbb{Z}^3\backslash\{0\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x}\right]. \tag{47}$$

We now construct our nonstationary constant-energy (for time $t \geqslant 0$) solution to the 3D Navier Stokes system on the torus using the results developed for the Stokes system. Recall that for any $j_0 \in \mathbb{Z}^2\backslash\{0\}$ there exists a sequence of indices $j_n = (j_n(1), j_n(2))$ such that $\lambda_{j_n} = 2^n\lambda_{j_0}$ for all $n \geqslant 0$.

Now consider the sequence $J_n \in \mathbb{Z}^3\backslash\{0\}$ defined such that $J_n = (j_n(1), j_n(2), 0)$. Notice that we have

$$\lambda_{\pm J_n} = \left(\frac{2\pi}{L}\right)^2 J_n \cdot J_n = \left(\frac{2\pi}{L}\right)^2 \left((j_n(1))^2 + (j_n(2))^2 + 0^2\right) = \lambda_{j_n}.$$

Thus, we have that $J_n$ so defined is such that $\lambda_{\pm J_n} = 2^n \lambda_{J_0}$ for $n \geqslant 0$.

First we define our forcing function $f(x) = \sum_{j\in\mathbb{Z}^3\backslash\{0\}} \hat{f}_j e^{i(2\pi/L)j\cdot x}$ as follows. Let $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, 0, f_{j_n}) = (0, 0, 1/b^n)$ for $b > \sqrt{2}$, and $\hat{f}_j = 0$ for all other $j \in \mathbb{Z}^3\backslash\{0\}$. By definition of the $\hat{f}_{J_n}$ terms we have that $f$ satisfies the reality condition. Note also that $f$ satisfies the divergence-free condition as well since $\hat{f}_{J_n} \cdot J_n = (0, 0, f_{j_n}) \cdot (j_n(1), j_n(2), 0) = 0$, and since $\hat{f}_j = 0$ (and thus $\hat{f}_j \cdot j = 0$) for all other $j$.

Let $u(x, t)$ be defined such that $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, 0, u_{j_n}(t))$, where $u_{j_n}(t)$ is from the nonstationary constant-energy solution of the Stokes system (for $t \geqslant 0$). Suppose further that $\hat{u}_j = 0$ for all other $j \in \mathbb{Z}^3\backslash\{0\}$. Then, as with $f$, $u$ satisfies the reality and divergence-free conditions.

We also have that $B(u, u) = 0$ for this choice of $u$. To see why, consider the sum

$$\sum_{j,k\in\mathbb{Z}^3\backslash\{0\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x}$$

from equation (47). Notice that for $j \notin \{\pm J_n\}_{n=0}^{\infty}$ we have that $\hat{u}_j = 0$ and thus $(\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} = 0$ for all $k$. Similarly, if $k \notin \{\pm J_n\}_{n=0}^{\infty}$ we have that $\hat{u}_k = 0$ and thus $(\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} = 0$ for all $j$. Thus equation (47) reduces to

$$B(u, u) = \mathcal{P}_L \left[ \frac{i2\pi}{L} \sum_{j,k\in\{\pm J_n\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} \right]. \tag{48}$$

However, if $j, k \in \{\pm J_n\}_{n=1}^{\infty}$, then $\hat{u}_j \cdot k = 0$ since $\hat{u}_j = (0, 0, u_j)$ and $k = (k(1), k(2), 0)$. Thus, we have that, for this definition of $u$, $B(u, u) = 0$.

Thus, this choice of $u$ puts us back in the Stokes system. We now need to confirm that this choice of $u$ satisfies $\frac{du}{dt} + \nu Au = f$ and that $u$ has constant energy for $t \geqslant 0$. Consider the following calculations:

$$\begin{aligned}
\frac{du}{dt} + \nu Au &= \sum_{j\in\mathbb{Z}^3\backslash\{0\}} \left(\frac{d}{dt} + \nu A\right) \hat{u}_j e^{i(2\pi/L)j\cdot x} \\
&= \sum_{n=1}^{\infty} \left(\frac{d}{dt} + \nu A\right) \hat{u}_{J_n} e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \left(\frac{d}{dt} + \nu A\right) \hat{u}_{-J_n} e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} \left(0, 0, u'_{j_n}(t) + \nu\lambda_{j_n} u(t)\right) e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \left(0, 0, u'_{j_n}(t) + \nu\lambda_{j_n} u(t)\right) e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} (0, 0, f_{j_n}) e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} (0, 0, f_{j_n}) e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} \hat{f}_{J_n} e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \hat{f}_{J_n} e^{i(2\pi/L)(-J_n)\cdot x} \\
&= f
\end{aligned}$$

where the move to the fourth line is due to the fact that the $u_{j_n}$ terms solve the Stokes system for those specific $f_{j_n}$ terms. Thus we have that the $u$ satisfies the 3D Navier-Stokes system. Now we confirm that the energy of $u$ is constant.

$$\begin{aligned}\frac{1}{2}|u(t)| &= \frac{L^3}{2}\sum_{j\in\mathbb{Z}^3\backslash\{0\}}|\hat{u}_j(t)|^2\\ &= \frac{L^3}{2}\left(\sum_{n=1}^{\infty}\lambda_{J_n}|\hat{u}_{J_n}(t)|^2+\sum_{n=1}^{\infty}|\hat{u}_{-J_n}(t)|^2\right)\\ &= \frac{L^3}{2}\left(\sum_{n=1}^{\infty}(u_{j_n}(t))^2+\sum_{n=1}^{\infty}(u_{j_n}(t))^2\right)\\ &= L^3\sum_{n=1}^{\infty}\left(\frac{f_{j_n}}{\nu\lambda_{j_n}}\right)^2,\end{aligned}$$

which is constant. The move to the last line is again justified by how the $u_{j_n}$ and $f_{j_n}$ terms were defined for the Stokes system. Straightforward calculations show that $u, f \in V$. Thus we have the following theorem:

**Theorem 4.15.** *Let $\{j_n\}_{n=0}^{\infty}$ and $f_j, u_j(0), u_j(t)$ for $j \in \mathbb{Z}^2\backslash\{0\}$ be defined as in Theorem 4.8. Let $\Omega = [0, L]^3_{per}$ and define $J_n = (j_n(1), j_n(2), 0)$. Define $\hat{f}_j$ such that $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, 0, f_{j_n})$ and $\hat{f}_j = 0$ for all other $j \in \mathbb{Z}^3\backslash\{0\}$. Define $\hat{u}_j(t)$ be such that $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, 0, u_{j_n}(t))$ and $\hat{u}_j(t) = 0$ for all other $j \in \mathbb{Z}^3\backslash\{0\}$. Then the function $u(x, t) = \sum_{j\in\mathbb{Z}^3\backslash\{0\}} \hat{u}_j(t)e^{i(2\pi/L)j\cdot x}$ is a nonstationary constant-energy solution to the Navier-Stokes system with force $f(x) = \sum_{j\in\mathbb{Z}^3\backslash\{0\}} \hat{f}_j e^{i(2\pi/L)j\cdot x}$ for $t \geqslant 0$.*

**Remark 4.16.** *The above construction lends itself just as easily to creating nonstationary solutions with constant norm $|A^{s/2}u|$, for any $s \in \mathbb{R}$, on the 3D periodic domain. Simply define the parameters as in Remark 4.9.*

**Remark 4.17.** *The above construction may be modified to create nonstationary constant-energy (or constant higher norm) solutions in dimension $d > 3$. Simply define the indices by $J_n = (j_n^{(1)}, j_n^{(2)}, 0, \cdots, 0)$ and the non-zero Fourier coefficients of $f$ and $u$ by $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, \cdots, 0, f_{j_n})$ and $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, \cdots, 0, u_{j_n}(t))$.*

**Remark 4.18.** *This construction cannot be directly applied to the 2D Navier-Stokes system. In the 3D case we were able to construct the non-zero Fourier coefficients $\hat{u}_{J_n}(t)$ and the indices $J_n$ such that they have disjoint support. This is what caused the nonlinear term to vanish and reduce the system to the Stokes system. This cannot be done in 2 dimensions since one cannot define the indices $J_n$ such that they are 0 in the second component and maintain the requirement that $J_n = 2^n\lambda_{J_0}$. This is because if $J_n = (J_n(1), 0)$, then $\lambda_{J_n} = \left(\frac{2\pi}{L}\right)^2 (J_n(1))^2$ is always a perfect square. However, $2^n\lambda_{J_0}$ would only be a perfect square when $n$ is even.*

**Remark 4.19.** *We recall the Reynolds number and Grashof number defined as follows: $Re = \frac{\langle u\rangle}{\nu\lambda_1^{1/2}}$ and $Gr = \frac{|f|}{\nu^2\lambda_1^{3/2}}$. We may also consider so-called "localized" Reynolds and Grashof numbers (localized to a specific mode) defined as $Re_j = \frac{|\hat{u}_j(0)|}{\nu\lambda_j^{1/2}}$ and $Gr_j = \frac{|\hat{f}_j|}{\nu^2\lambda_j^{3/2}}$. Recall that the requirement* *on the parameters for maintaining constant energy is $2\nu\lambda_{j_0}u_{j_0}(0)b^2 \leqslant 1$. Thus, we may write this requirement in terms of localized Reynolds and Grashof numbers as follows:*

$$\frac{Re_{j_0}}{Gr_{j_2}} < 4.$$

## 5. Finite-Mode Solutions on the 2D Torus

This section is dedicated to establishing Theorem 5.2 and its consequences.

5.1. **Spectral Structure Theorem.** We begin with a calculation that further simplifies the Fourier coefficient of the nonlinear term when $\Omega$ is the 2D torus.

**Lemma 5.1.** *The $j^{th}$ Fourier mode of the nonlinear term of the NSE on the 2D torus may be written as*

$$\widehat{B(u,u)}_j = \frac{i\pi}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)(|k|^2 - |m|^2)\frac{u_m u_k}{|j|^2} j^\perp \tag{49}$$

*where in general $u_\ell$ is a scalar in $\mathbb{C}$ defined such that $u_\ell \ell^\perp = \hat{u}_\ell$, with $\ell^\perp = (l_1, l_2)^\perp = (-l_2, l_1)$.*

*Proof.* Recall the explicit representation of the $j^{\text{th}}$ Fourier mode of the nonlinear term given in equation (18):

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L} \sum_{k\in\mathbb{Z}^2\backslash\{0\}} \frac{(\hat{u}_{j-k}\cdot k)(\hat{v}_k \cdot j^\perp)}{|j|^2} j^\perp.$$

Recall that the divergence-free condition on $u$ requires that $\hat{u}_j \cdot j = 0$ for all $j \in \mathbb{Z}^2\backslash\{0\}$. In 2D this means that we may write $\hat{u}_j = u_j j^\perp$ where $u_j$ is now a scalar in $\mathbb{C}$ and $j^\perp = (-j_2, j_1)$. Writing $u$ this way we write the $j^{\text{th}}$ Fourier coefficient of the $B(u,u)$ term as follows:

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{k\in\mathbb{Z}^2} ((j-k)^\perp \cdot k)(k^\perp \cdot j^\perp)\frac{u_{j-k}u_k}{|j|^2} j^\perp. \tag{50}$$

Note that without loss of generality we relaxed the requirement on the index from $k \in \mathbb{Z}^2\backslash\{0\}$ to $k \in \mathbb{Z}^2$ since an index of $k = (0,0)$ would not contribute to the sum anyway. By reindexing with $m = j - k$, the coefficient can be written as

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{\substack{k,m\in\mathbb{Z}^2 \\ k+m=j}} (m^\perp \cdot k)(k^\perp \cdot (m+k)^\perp)\frac{u_m u_k}{|j|^2} j^\perp. \tag{51}$$

A simple calculation shows that for $a, b \in \mathbb{Z}^2$ we have $a^\perp \cdot b^\perp = a \cdot b$, given how we've chosen to define the perpendicular vectors. Thus we may also write the nonlinear term coefficient as

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{\substack{k,m\in\mathbb{Z}^2 \\ k+m=j}} (m^\perp \cdot k)(k \cdot (m+k))\frac{u_m u_k}{|j|^2} j^\perp. \tag{52}$$

Consider the following calculation regarding the $m^{\text{th}}$ coefficient of the nonlinear term:

$$\begin{aligned}
2\widehat{B(u,u)}_j &= \frac{2\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}(m^\perp\cdot k)(k\cdot(m+k))\frac{u_m u_k}{|j|^2}j^\perp + \frac{2\pi i}{L}\sum_{\substack{k,m\in\mathbb{Z}^2\\ k+m=j}}(k^\perp\cdot m)(m\cdot(k+m))\frac{u_k u_m}{|j|^2}j^\perp \\
&= \frac{2\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}\left((m^\perp\cdot k)(k\cdot(m+k))\frac{u_m u_k}{|j|^2}j^\perp + (k^\perp\cdot m)(m\cdot(k+m))\frac{u_k u_m}{|j|^2}j^\perp\right) \\
&= \frac{2\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}\left((m^\perp\cdot k)(k\cdot(m+k))\frac{u_m u_k}{|j|^2}j^\perp - (m^\perp\cdot k)(m\cdot(k+m))\frac{u_k u_m}{|j|^2}j^\perp\right) \\
&= \frac{2\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}(m^\perp\cdot k)((k-m)\cdot(m+k))\frac{u_m u_k}{|j|^2}j^\perp \\
&= \frac{2\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}(m^\perp\cdot k)(|k|^2-|m|^2)\frac{u_m u_k}{|j|^2}j^\perp
\end{aligned}$$

Thus we may write

$$\widehat{B(u,u)}_j = \frac{\pi i}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}(m^\perp\cdot k)(|k|^2-|m|^2)\frac{u_m u_k}{|j|^2}j^\perp. \tag{53}$$

□

Notice that if wavenumbers $m, k$ are such that $m$ is parallel to $k$ then $m^\perp\cdot k = 0$, and so that term in the sum is 0. Notice also that if $m, k$ are such that $|m| = |k|$ then that term in the sum is also 0. Thus, the only pairs of wave numbers that contribute to the nonlinear term in the $j^{\text{th}}$ mode are pairs of different length, that are not parallel, and whose sum is equal to $j$. Finally note that each wavenumber in such a pair must correspond to a Fourier mode where $u$ is supported in order to contribute to the nonlinear term.

Thus the expression of the NSE in terms of Fourier modes becomes

$$\hat{u}_j' + \nu\lambda_j\hat{u}_j + \frac{i\pi}{L}\sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}}(m^\perp\cdot k)(|k|^2-|m|^2)\frac{u_m u_k}{|j|^2}j^\perp = \hat{f}_j; \qquad j\in\mathbb{Z}^2\backslash\{0\}.$$

**Theorem 5.2.** *Let* $u(x,t) = \sum_{\substack{j\in\mathbb{Z}^2\backslash\{0\}\\ |j|\leq N}}\hat{u}_j(t)e^{ij\cdot x}$ *be a finite-mode solution to the NSE on the 2D torus with nontrivial nonlinear term (i.e.* $B(u,u)\neq 0$*). Then there exist wavenumbers* $j, k$ *such that the following criteria hold:*

*(1)* $\hat{u}_{j+k} = 0$,

*(2)* $\widehat{B(u,u)}_{j+k} = \frac{i\pi}{L}(j^\perp\cdot k)(|k|^2-|j|^2)\frac{u_j u_k}{|j+k|^2}(j+k)^\perp$

*(3)* $\widehat{B(u,u)}_{j+k}\neq 0$

*Proof.* Suppose $u$ is a finite-mode solution to the NSE with nontrivial nonlinear term. Let $S$ be the finite set of vectors in $\mathbb{Z}^2\backslash\{0\}$ associated with the Fourier modes where $u$ is supported. That is, $j \in S$ if and only if $\hat{u}_j \neq 0$. We seek a pair of wave numbers $j, k \in \mathbb{Z}^2\backslash\{0\}$ with the following properties:

(1) $j, k \in S$

(2) $j + k \notin S$

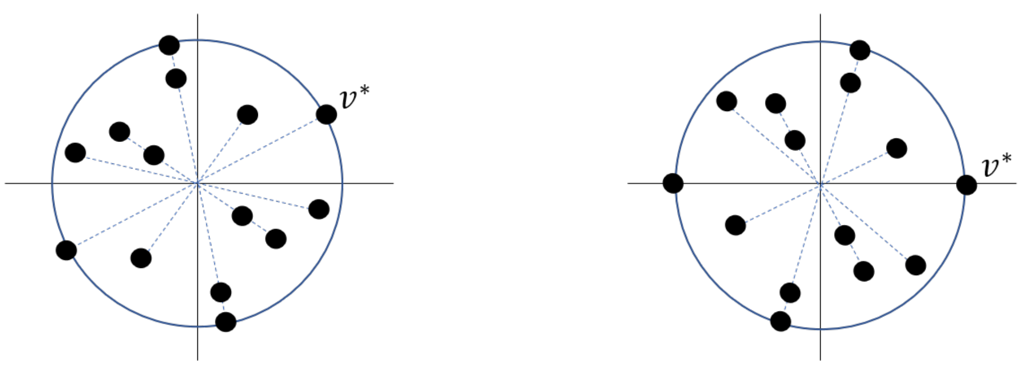
(3) $j \nparallel k$

(4) $|j| \neq |k|$

(5) If $p, q \in S$ are such that $\{p, q\} \neq \{j, k\}$ and $p + q = j + k$ then either $p \parallel q$ or $|p| = |q|$

Property 2 is there to guarantee that the first criterion of our theorem is satisfied. Properties 1, 3, and 4 together imply that $\frac{i\pi}{L}(j^\perp \cdot k)(|k|^2 - |j|^2)\frac{u_j u_k}{|j+k|^2}(j+k)^\perp \neq 0$ and contributes to the sum for the $(j+k)^{\text{th}}$ mode of nonlinear term. (We recall that the Fourier coefficients of any solution to the NSE are analytic in time. This implies that if $u_j(t)$ and $u_k(t)$ are not identically zero, then the product $u_j(t)u_k(t)$ is not identically zero, since nonzero analytic functions may only take the value of 0 on a discrete set of points.) Property 5 implies that for any pair of vectors $p, q \in S$ such that $\{p, q\} \neq \{j, k\}$ we have $\frac{i\pi}{L}(p^\perp \cdot q)(|p|^2 - |q|^2)\frac{u_p u_q}{|p+q|^2}(p+q)^\perp = 0$ and does not contribute to the sum for the $(j+k)^{\text{th}}$ mode of nonlinear term. Properties 1, 3, 4, and 5 together establish the second and third criteria in the theorem.

We begin by choosing a vector $v^* \in S$ such that for any $v_i \in S$ we have $|v_i| \leqslant |v^*|$ (*i.e.* $v^*$ has maximum length in $S$). We let $b_1 = v^*$ and $b_2 = (v^*)^\perp$ be a new basis for our vector space and we write the vectors in $S$ in the coordinates of this new basis. Geometrically speaking, we reorient the plane so that $v^*$ lies on the $x$-axis. [See Figure 1]




(A) Original orientation of vectors in $S$ (B) Reorientation of vectors in $S$

Figure 1

We then arrange the elements of $S$ in reverse-lexicographic order (according to this new basis). That is, $j$ appears earlier in our list than $k$ if and only if either the first component of $j$ is greater than the first component of $k$ (that is, $j$ is farther along in the direction of $v^*$ than $k$), or if the first components are equal then the second component of $j$ is greater than the second component of $k$ (that is, $j$ is farther along in the direction $90^\circ$ counterclockwise from $v^*$ than $k$). Assuming $S$ contains $n$ vectors, let us relabel the vectors in $S$ as $v_1, v_2, \ldots, v_n$ according to their reverse-lexicographic order. Thinking in terms of Figure 1b, the vectors are ordered from right to left (with $v_1 = v^*$), and if a set of vectors are equally far right, these vectors are ordered from top to bottom.

Recall that the Fourier coefficients of $u$ come in pairs. That is, if $v_j$ is in $S$ then so is $-v_j$. This implies that $n$ is an even number, and, given our ordering, the vectors $v_1, \ldots, v_{n/2}$ have nonnegative first components. Also notice that since we assume $B(u,u) \neq 0$, we have that $n \geqslant 4$.

Consider the vector $v_1 + v_2$. [See Figure 2]. We can see $v_1 + v_2 \notin S$ since if $v_2$ has positive first component then the first component of $v_1 + v_2$ is greater than the first component of $v_1$, which has the largest first component of any vector in $S$. If the first component of $v_2$ is 0 then the second component is positive and this implies that $v_1 + v_2$ has greater length than $v_1$, which already has length greater than or equal to any vector in $S$. Thus $v_1$ and $v_2$ are wavenumbers that satisfy Properties 1 and 2 of our list.

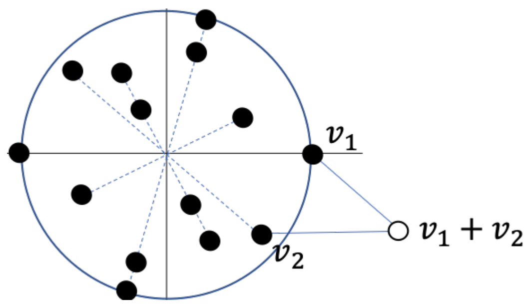



FIGURE 2. Vector $v_1 + v_2$

To establish Property 5 we show that if $v_p, v_q \in S$ are such that $v_p + v_q = v_1 + v_2$ then $\{v_p, v_q\} = \{v_1, v_2\}$. The proof of this straightforward. The geometric idea is that any other pair of vectors would have a sum that does not extend as far up and to the right as (*i.e.* has a smaller first or second component than) $v_1 + v_2$. Suppose $v_p, v_q \in S$ are such that $v_p + v_q = v_1 + v_2$. Note that if $v_p \in \{v_1, v_2\}$ or if $v_q \in \{v_1, v_2\}$ then $v_p + v_q = v_1 + v_2$ implies that $\{v_p, v_q\} = \{v_1, v_2\}$. So suppose $v_p, v_q \notin \{v_1, v_2\}$. Denote the $i^{\text{th}}$ component of a vector $v$ in the basis $\{b_1, b_2\}$ by $v(i)$. Note that, given the ordering of vectors in $S$, for any $v \in S$ with $v \neq v_1$ we have that $v(1) \leqslant v_2(1) < v_1(1)$. Thus we have $v_p(1), v_q(1) \leqslant v_2(1) < v_1(1)$ and so $v_p(1) + v_q(1) < v_1(1) + v_2(1)$ and $v_p + v_q \neq v_1 + v_2$.

Thus $v_1$ and $v_2$ are wavenumbers that satisfy Property 5. If we also have that $v_1 \nparallel v_2$ and if $|v_1| \neq |v_2|$ then Properties 3 and 4 are satisfied and $v_1$ and $v_2$ are the required vectors for the theorem (This case is represented in Figure 2.)

However, if $v_1 \parallel v_2$ or $|v_1| = |v_2|$ then $v_1$ and $v_2$ do not satisfy the theorem and we must seek a different pair of wavenumbers that satisfy the five properties listed above. We handle these cases in turn.

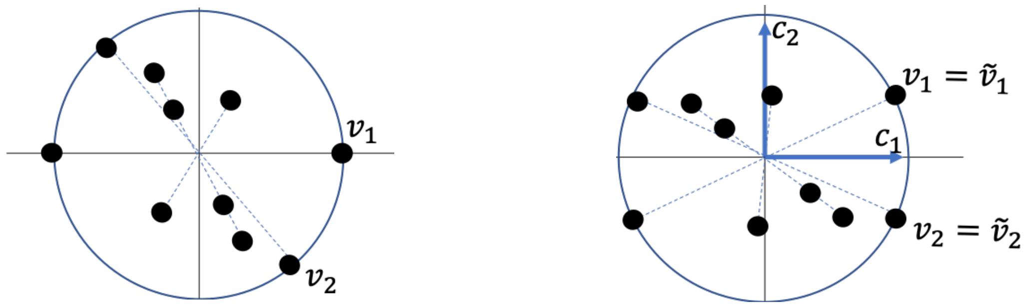
**Case 1:** $|v_1| = |v_2|$

For this case we again define a new basis for our vector space. Let $c_1 = \frac{v_1+v_2}{2}$ and $c_2 = \frac{(v_1+v_2)^\perp}{2}$ be our new basis vectors (see Figure 3).[1]

In this case the vectors $v_1$ and $v_2$ have the same first coordinate in the basis $C = \{c_1, c_2\}$, and that first coordinate is strictly larger than the first coordinate (in $C$) of any other vector in $S$. (If any other vector had the same or larger first coordinate in $C$ then that vector would have been come between $v_1$ and $v_2$ in the ordering under the basis $\{b_1, b_2\}$, which is a contradiction.) Let $\tilde{v}_1 \ldots, \tilde{v}_n$

[1]Why choose $c_1 = \frac{v_1+v_2}{2}$ and $c_2 = \frac{(v_1+v_2)^\perp}{2}$ as opposed to, say, $c_1 = v_1 + v_2$ and $c_2 = (v_1 + v_2)^\perp$ or $c_1 = \frac{v_1+v_2}{|v_1+v_2|}$ and $c_2 = \frac{(v_1+v_{j_0})^\perp}{|v_1+v_2|}$? The choice is nothing deep. The problem with $v_1 + v_2$ is that such a vector will not fit compactly in our diagrams. The problem with $\frac{v_1+v_2}{|v_1+v_2|}$ is that it is more cumbersome to type.

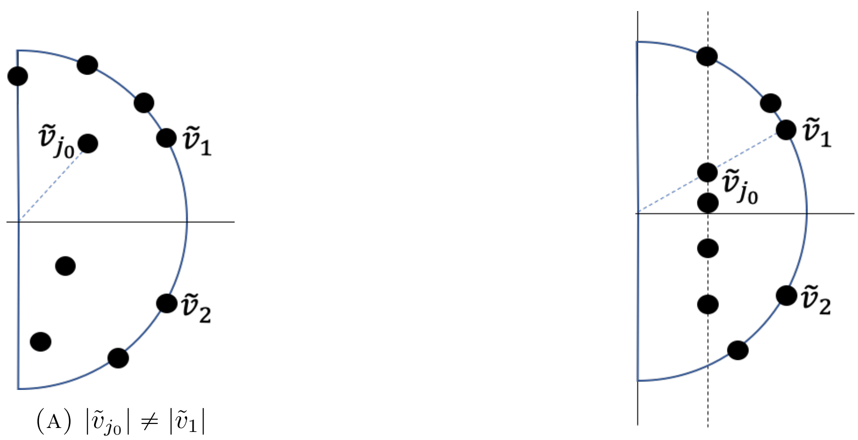
be the reverse-lexicographic ordering of the vectors in $S$ according to the basis $C$. Without loss of generality we may assume $\tilde{v}_1 = v_1$ and $\tilde{v}_2 = v_2$ as in Figure 3. Note that since that by construction, in basis $C$, $\tilde{v}_1(1) = \tilde{v}_2(1) > 0$.




(A) Original orientation (B) Reoriented

FIGURE 3. $|v_1| = |v_2|$

Let $\tilde{v}_{j_0}$ be the first vector in this new ordering of $S$ such that $|\tilde{v}_{j_0}| \neq |\tilde{v}_1|$ (see Figure 4a for example). If no such vector exists then $B(u, u) = 0$ and we are outside the scope of our theorem. We are now in a position to find pairs of vectors that satisfy the required five properties. Again, we must consider cases:




(A) $|\tilde{v}_{j_0}| \neq |\tilde{v}_1|$ (B) $|\tilde{v}_{j_0}| \neq |\tilde{v}_1|$, $\tilde{v}_{j_0} \parallel \tilde{v}_1$

FIGURE 4. $|\tilde{v}_{j_0}| \neq |\tilde{v}_1|$

(1.a) $\tilde{v}_{j_0} \nparallel \tilde{v}_1$

In this case, the pair $\tilde{v}_1, \tilde{v}_{j_0}$ by definition satisfy Properties 1, 3, and 4. To see that $\tilde{v}_{j_0} + \tilde{v}_1 \notin S$ note that either $\tilde{v}_{j_0}$ has a positive first component (in the basis $C = \{c_1, c_2\}$), or $\tilde{v}_{j_0}$ has 0 as its first component and has a positive second component. In the first case $\tilde{v}_{j_0} + \tilde{v}_1$ has a larger first component than $\tilde{v}_1$, and thus is not in $S$. In the second case $|\tilde{v}_{j_0} + \tilde{v}_1| > |\tilde{v}_1|$ and thus is not in $S$. Thus Property 2 is satisfied.

To see that Property 5 is satisfied we consider a pair of vectors $\tilde{v}_p, \tilde{v}_q \in S$ such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ with $\{\tilde{v}_p, \tilde{v}_q\} \neq \{\tilde{v}_1, \tilde{v}_1\}$. Without loss of generality, suppose $p < q$ (that is, $\tilde{v}_p$ comes before $\tilde{v}_q$ in our reverse-lexicographic ordering in basis $C$). We show that any such pair of vectors will be such that $1 < p < q < j_0$ and thus $\tilde{v}_p, \tilde{v}_q$ do not contribute to $B(u, u)$, since $|\tilde{v}_p| = |\tilde{v}_q|$ (since $p, q < j_0$ implies that both $\tilde{v}_p$ and $\tilde{v}_q$ have the same length as $\tilde{v}_1$). Recall that, given the ordering under the basis $C$, for any $\tilde{v} \in S$ such that $\tilde{v} \notin \{\tilde{v}_1, \tilde{v}_2\}$ we have that $\tilde{v}(1) < \tilde{v}_1(1) = \tilde{v}_2(1)$.

As we saw previously, if any other pair of vectors $\tilde{v}_p, \tilde{v}_q$ is such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ then neither $\tilde{v}_p$ nor $\tilde{v}_q$ is equal to $\tilde{v}_1$ or $\tilde{v}_{j_0}$.

Suppose $\tilde{v}_p = \tilde{v}_2$. Then $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ implies that, in the basis $C$, $\tilde{v}_q(1) = \tilde{v}_{j_0}(1)$ (since $\tilde{v}_p(1) = \tilde{v}_2(1) = \tilde{v}_1(1)$). This also implies $\tilde{v}_q(2) > \tilde{v}_{j_0}(2)$ (since $\tilde{v}_p(2) = \tilde{v}_2(2) < \tilde{v}_1(1)$). This implies that $q < j_0$ and thus that $1 < p < q < j_0$. Thus, by our choice of $j_0$, $|\tilde{v}_p| = |\tilde{v}_q| = |\tilde{v}_1|$ as desired.

Suppose $\tilde{v}_p \neq \tilde{v}_2$, i.e. $p > 2$. Then we have, in particular, that $\tilde{v}_p(1) < \tilde{v}_1(1)$. Thus $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ implies that $\tilde{v}_{j_0}(1) < \tilde{v}_q(1)$. This implies that $j_0 > q$ and thus that $1 < p < q < j_0$. Thus $|\tilde{v}_p| = |\tilde{v}_q| = |\tilde{v}_1|$ as desired.

Therefore, we conclude that the vectors $\tilde{v}_{j_0}$ and $\tilde{v}_1$ are vectors that satisfy the theorem.

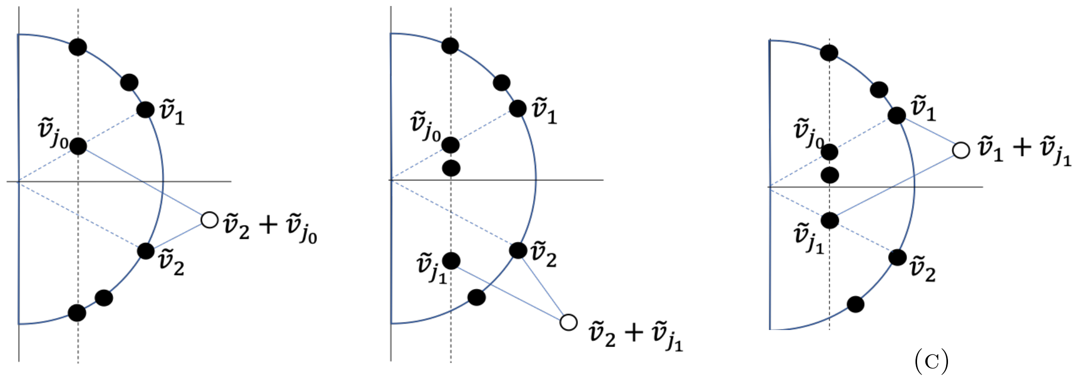
(1.b) $\tilde{v}_{j_0} \parallel \tilde{v}_1$

In this case consider the set of vectors in $S$ with the same first component (in the basis $C$) as $\tilde{v}_{j_0}$ and with size strictly less than $|\tilde{v}_1|$. Call this set of vectors $S'$. ($S'$ is the set of vectors in Figure 4b that lie on the vertical dotted line, but not on the outer circle).

- If $\tilde{v}_{j_0}$ is the only vector in this set (*i.e.* $S' = \{\tilde{v}_{j_0}\}$) then the pair $\tilde{v}_{j_0}, \tilde{v}_2$ satisfies all the conditions of the theorem. This is again because $\tilde{v}_2(1) + \tilde{v}_{j_0}(1) > \tilde{v}_1(1)$ (and thus $\tilde{v}_2 + \tilde{v}_{j_0} \notin S$), and any other pair of vectors $\tilde{v}_p, \tilde{v}_q$ such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_2 + \tilde{v}_{j_0}$ are such that $|\tilde{v}_p| = |\tilde{v}_q|$, and the reasoning is similar to the case (1.a). See Figure 5a.
- If there are other vectors in $S'$, then let $\tilde{v}_{j_1}$ be the vector in $S'$ with the smallest second coordinate. If $\tilde{v}_{j_1}$ is not parallel to $\tilde{v}_2$ (see Figure 5b), then the pair $\tilde{v}_{j_1}, \tilde{v}_2$ satisfies all of the conditions of our theorem. Essentially, this is because any other pair of vectors whose sum matches the first coordinate of $\tilde{v}_{j_1} + \tilde{v}_2$ must either both have maximum length (which is what we want) or one is from $S'$ and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$. However, since $\tilde{v}_{j_1}$ has the smallest second coordinate of any vector in $S'$ and $\tilde{v}_2$ has a smaller second coordinate than $\tilde{v}_1$, no other sum of a vector from $S'$ and a vector from $\{\tilde{v}_1, \tilde{v}_2\}$ will match the second coordinate of $\tilde{v}_{j_1} + \tilde{v}_2$.
- If both $\tilde{v}_{j_0} \parallel \tilde{v}_1$ and $\tilde{v}_{j_1} \parallel \tilde{v}_2$ (see Figure 5c), then the pair $\tilde{v}_{j_1}, \tilde{v}_1$ satisfies the theorem. As before, this is because any other pair of vectors whose sum matches the first coordinate of $\tilde{v}_{j_1} + \tilde{v}_1$ must either both have maximum length or one is from $S'$ and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$. Note that in our current case, any vector from $S'$ added to $\tilde{v}_1$ will have a positive second coordinate and any vector from $S'$ added to $\tilde{v}_2$ will have a negative second component. This is due to the fact that $\tilde{v}_2(2) = -\tilde{v}_1(2)$ and for any $\tilde{v}_i \in S'$ we have $|\tilde{v}_i(2)| < |\tilde{v}_1(2)|$ (owing to the fact that the vectors from $S'$ now are sandwiched between two vectors, $\tilde{v}_{j_0}$ and $\tilde{v}_{j_1}$, that are positive scalar multiples of $\tilde{v}_1$ and $\tilde{v}_2$ where that scalar is strictly less than 1). Thus, of all the sums of vectors where one is from $S'$ and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$, the sum $\tilde{v}_{j_1} + \tilde{v}_1$ is the only sum having the smallest positive second component. Thus, the pair $\tilde{v}_{j_1}, \tilde{v}_1$ satisfy the conditions of our theorem. Indeed, in this case, any vector from $S' \backslash \{\tilde{v}_{j_0}\}$ together with $\tilde{v}_1$ will meet the conditions of the theorem (as well as any vector from $S' \backslash \{\tilde{v}_{j_1}\}$ together with $\tilde{v}_2$).

Thus we have shown that even in the case where $|v_1| = |v_2|$ we can find wavenumbers $j, k$ that satisfy all the conditions of the theorem.

**Case 2:** $v_1 \parallel v_2$




(A) (B) (C)

FIGURE 5. $\tilde{v}_{j_0} \parallel \tilde{v}_1$

If $v_2$ is parallel to $v_1$, let $v_{k_0}$ be the first ordered (in basis $\{b_1, b_2\}$) vector from $S$ such that $v_{k_0}$ is not parallel to $v_1$. If no such vector exists then $B(u, u) = 0$ and we are outside the scope of our theorem. Thus we may assume that such a vector exists. Note that such a vector must have a nonnegative first component. (Otherwise we would have $-v_{k_0}$ precede $v_{k_0}$ on the list. But we already said that any vector preceding $v_{k_0}$ was parallel to $v_1$, implying that $-v_{k_0}$ and thus $v_{k_0}$ was parallel to $v_1$. This is contrary to our assumption on $k_0$). We must consider two further subcases. The first is when $|v_{k_0}| \neq |v_1|$. The second is when $|v_{k_0}| = |v_1|$.

(2.a) $|v_{k_0}| \neq |v_1|$:

In this case the pair $v_1, v_{k_0}$ by definition satisfies Properties 1, 3, and 4. We consider the vector $v_1 + v_{k_0}$ (see Figure 6). To see that $v_1 + v_{k_0} \notin S$ note that either $v_{k_0}$ has a positive first component, or $v_{k_0}$ has 0 as its first component and has a positive second component. In the first case $v_{k_0} + v_1$ has a larger first component than $v_1$, and thus is not in $S$. In the second case $|v_{k_0} + v_1| > |v_1|$ and thus is not in $S$. Thus Property 2 is satisfied.

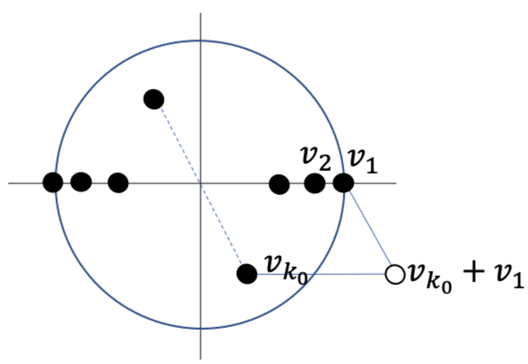



FIGURE 6. Vector $v_1 + v_{k_0}$

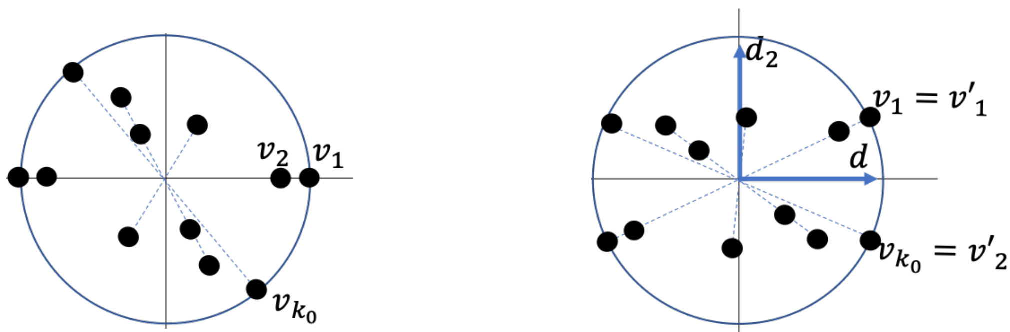
To see that Property 5 is satisfied we consider a pair of vectors $v_p, v_q \in S$ such that $v_p + v_q = v_1 + v_{k_0}$. We show that any such pair of vectors such that $\{v_p, v_q\} \neq \{v_1, v_{k_0}\}$ will also be such that $1 < p < q < k_0$ and thus that $v_p \parallel v_q$ (since $p, q < k_0$ implies that both $v_p$ and $v_q$ are parallel to $v_1$) and therefore do not contribute to $B(u, u)$.

As we saw above, if one of $v_p, v_q \in \{v_1, v_{k_0}\}$ then $\{v_p, v_q\} = \{v_1, v_{k_0}\}$. So assume that neither $v_p$ nor $v_q$ is equal to $v_1$ or $v_{k_0}$. Note that since for any $v \in S \backslash \{v_1\}$ we have $v(1) < v_1(1)$, the identity $v_p + v_q = v_1 + v_{k_0}$ implies that $v_p(1), v_q(1) > v_{k_0}$. This implies that $1 < p, q < k_0$, and thus that $v_p \parallel v_q$. (Note that in fact there are no vectors $v_p, v_q \in S$ such that $v_p + v_q = v_1 + v_{k_0}$ and $\{v_p, v_q\} \neq \{v_1, v_{k_0}\}$.)

Thus the conditions of the theorem are satisfied in this case.

(2.b) $|v_{k_0}| = |v_1|$:

For this case we redefine a basis for our vector space in a similar manner to what we did in Case 1. Let $d_1 = \frac{v_1 + v_{k_0}}{2}$ and $d_2 = \frac{(v_1 + v_{k_0})^{\perp}}{2}$ be our new basis vectors (see Figure 7).




(A) $|v_{k_0}| = |v_1|$

(B) Reoriented

FIGURE 7. $|v_{k_0}| = |v_1|$

In this case the vectors $v_1$ and $v_{k_0}$ have the same first coordinate in the basis $D = \{d_1, d_2\}$, and that first coordinate is strictly larger than the first coordinate (in $D$) of any other vector in $S$. (If any other vector had the same or larger first coordinate in $D$ then that vector would have come between $v_1$ and $v_{k_0}$ in the ordering under the basis $\{b_1, b_2\}$. But by assumption the only such vectors are parallel to $v_1$. Thus each such vector is of the form $v_i = c_i v_1$ where $c_i < 1$, and so each such vector has smaller first coordinate than $v_1$ in any basis.) Let $v'_1, \ldots, v'_n$ be the reverse-lexicographic ordering of the vectors in $S$ according to the basis $D$. Without loss of generality we may assume $v'_1 = v_1$ and $v'_2 = v_{k_0}$. (See Figure 7b)

Notice now that we are in the same position we were in at the beginning of Case 1. Thus, we may treat this situation in the same way as we treated Case 1.

□

5.2. **Allowable Forces for Finite Mode Solutions.** This section is dedicated to establishing limits on the types of forces that can admit of the possibility of a finite mode solution to the 2D NSE with non-trivial non-linear term. As a consequence, we will show that no finite-mode solution with nontrivial nonlinear term is possible in the case when the force is an eigenvector of the Stokes operator. This, combined with results from Section 4, proves the impossibility of so-called *chained ghost solutions* introduced in [24].

We begin with the following corollary to Theorem 5.2.

**Corollary 5.3.** *Let $u$ be a finite mode solution to the NSE on the 2D torus with non-trivial non-linear term. Let $S_f$ (resp. $S_u$) be the set of wave numbers associated with the Fourier modes where $f$ (resp. $u$) is supported. For all pairs of wavenumbers $j, k \in S_u$ that satisfy the conditions of Theorem 5.2, $f$ must be supported on the Fourier mode associated with $j + k$ (i.e. $j + k \in S_f$).*

*In particular, let $k_f \in S_f$ be such that $|k_f| \geqslant k$ for all $k \in S_f$ and let $k_u \in S_u$ be such that $|k_u| \geqslant k$ for all $k \in S_u$. Then $|k_f| > |k_u|$.*

*Proof.* Recall the following Fourier characterization of the NSE:

$$\hat{u}_j' + \nu\lambda_j \hat{u}_j + \widehat{B(u,u)}_j = \hat{f}_j; \qquad j \in \mathbb{Z}^2 \backslash \{0\}. \tag{54}$$

Suppose $u$ is a finite mode solution to the NSE. Let $j, k$ be a pair of wave numbers that satisfy the conditions of the Theorem 5.2. Then the Fourier characterization of the NSE for the mode associated with wavenumber $j+k$ is as follows:

$$\widehat{B(u,u)}_{j+k} = \hat{f}_{j+k} \tag{55}$$

Since $\widehat{B(u,u)}_{j+k} \neq 0$ by Theorem 5.2, this implies that $\hat{f}_{j+k} \neq 0$. In the proof of Theorem 5.2 it is demonstrated that $j$ and $k$ can be chosen such that $|j+k| > k_u$ for any $k_u \in S_u$. □

Note that Corollary above implies that any nontrivial (in the sense that $B(u,u) \neq 0$) finite-mode solution must have a force $f$ that is supported outside the spectrum of the solution $u$, in particular $f$ must have modes strictly bigger then the largest non-zero mode in the solution, $|k_u|$. In the trivial case, i.e. $B(u,u) = 0$ (for example if a solution lives in an eigenspace of $A$ or if the non-zero wavenumbers in a solution are parallel), the Fourier components become decoupled, and therefore, any trivial finite-mode solution on the global attractor is necessarily a steady state. In the case of nontrivial solutions we obtain the following result.

**Theorem 5.4.** *In the system (6) let $\Omega = [0, L]^2_{per}$ and let the force be an eigenvector of the Stokes operator. Let $u$ be a solution to (6) such that $B(u,u) \neq 0$. Then $u$ must be supported on an infinite number of Fourier modes.*

*Proof.* Let the force $f$ be an eigenvector of the Stokes operator. Assume that $u$ is a solution to the NSE with force $f$ such that $B(u,u) \neq 0$ and $u$ is supported on a finite number of Fourier modes. Let $S_f$ (resp. $S_u$) be the set of wave numbers associated with the Fourier modes where $f$ (resp. $u$) is supported. Since $f$ is an eigenvector of the Stokes operator it must be the case that for any $k_{f1}, k_{f2} \in S_f$ we have $|k_{f1}| = |k_{f2}|$. By Corollary 5.3 this implies that for any $k_f \in S_f$ and $k_u \in S_u$ we have $|k_f| > |k_u|$. This implies that $(f, u) = 0$.

The energy balance equation in this case is simply:

$$\frac{1}{2}\frac{d}{dt}|u|^2 = -\nu\|u\|^2. \tag{56}$$

By the Poincaré inequality this yields $\frac{1}{2}\frac{d}{dt}|u|^2 \leqslant -\nu\lambda_0|u|^2$. Applying Gronwall's inequality we have $|u(t)|^2 \leqslant |u(0)|^2 e^{-2\nu\lambda_0 t}$. Thus we have that $\lim_{t\to\infty}|u(t)| \to 0$ and thus $u(t)$ converges to 0 in $H$. Moreover, since the solution operator $S(t)$ depends continuously on the data, 0 must be a fixed point (stationary solution) on the global attractor, which implies that $f \equiv 0$. This contradicts our assumption on $f$. Thus any solution $u$ with nontrivial nonlinear term must be supported on an infinite number of Fourier modes.

□

**Corollary 5.5.** *When the force is an eigenvector the only finite-mode solution on the global attractor of to the 2D NSE is the trivial solution $u = \frac{f}{\nu\lambda_f}$ (where $\lambda_f$ is the eigenvalue associated with the eigenvector $f$). In particular, $u = \frac{f}{\nu\lambda_f}$ is the only finite-mode stationary solution.*

*Proof.* By Theorem 5.4, if a solution $u$ is such that $B(u,u) \neq 0$ then $u$ cannot be finite-mode. In the case where $B(u,u) = 0$ we are in the Stokes system (24), where the only bounded solution is the trivial solution $u = \frac{f}{\nu\lambda_f}$.

□

**Remark 5.6.** *Looking outside the global attractor, we note that the proof of the Theorem 5.4 implies that in the case the force $f$ is an eigenvector corresponding to an eigenvalue $\lambda_f$, the only finite-mode solutions to the Navier-Stokes equations are either those where the non-zero modes are parallel to the modes of $f$ (this can only happen if $f$ is a one-mode force) or those where the modes are equal in magnitudes to the modes of $f$. In both cases such solutions converge as $t \to \infty$ to the stationary solution $u = \frac{f}{\nu\lambda_f}$.*

**Corollary 5.7.** *There are no finite-mode ghost solutions to the 2D NSE when the force is an eigenvector of the Stokes operator. In particular, there are no so-called chained ghost solutions.*

*Proof.* Theorem 5.4 demonstrates that no finite-mode solution exists at all when the force is an eigenvector and the solution has nontrivial nonlinear term. In the case where the nonlinear term is trivial we are in the Stokes system. Theorem 4.12 implies that there are no ghost solutions possible in the Stokes system. Thus, when the force is an eigenvector of the Stokes operator, there do not exist any finite-mode ghost solutions. In particular, this implies that there do not exist chained ghost solutions. □

Naively, there is nothing to rule out the possibility of a finite-mode solution so long as the force is supported on all wavenumbers $k$ where $\hat{u}_k = 0$ but $\widehat{B(u,u)}_k \neq 0$. However, for wavenumbers $j$ and $k$ where the conditions of Theorem 5.2 are satisfied, we have an interesting condition on the Fourier coefficients $\hat{u}_j$ and $\hat{u}_k$.

**Proposition 5.8.** *Let $j, k$ be wavenumbers that satisfy the conditions of Theorem 5.2. Then the product $u_j u_k$ is constant, and both $u_j$ and $u_k$ are nowhere vanishing.*

*Proof.* When $j, k$ satisfy the condition of Theorem 5.2, the Fourier characterization of the $(j+k)^{\text{th}}$ mode of the NSE can be written as $c_{j,k} u_j(t) u_k(t) = f_{j+k}$, where $c_{j,k} = (j^\perp \cdot k)(k \cdot (j+k)) \frac{1}{|m|^2}$ is (a non-zero) constant. Since $f_k$ is also (a nonzero) constant, this implies that the product $u_j(t)u_k(t)$ is constant. It is well-known that the functions $u_j(t)$ and $u_k(t)$ are analytic in time. Thus, in order for the product to be a non-zero constant, we must have that neither function is ever equal to 0. □

## 6. Appendix

Define our domain $\Omega$ as follows: $\Omega = [0, L]^n_{per}$ for $n = 2, 3$. First we represent $(u \cdot \nabla)v$. Any divergence-free $L^2$ function on the torus may be represented by $u(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{u}_k e^{i(2\pi/L)k \cdot x}$, where each $\hat{u}_k$ is a vector in $\mathbb{C}^n$ such that $\overline{\hat{u}_k} = \hat{u}_{-k}$, $\hat{u}_k \cdot k = 0$, and $\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}_k|^2 < \infty$. Consider the following formal calculations, which can be made rigorous by assuming $u, v \in D(A)$.

$$\begin{aligned}(u\cdot\nabla)v &= (u\cdot\nabla)\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{v}_k e^{i(2\pi/L)k\cdot x}\\ &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{v}_k\sum_{j=1}^{n}u_j\frac{\partial}{\partial x_j}e^{i(2\pi/L)k\cdot x}\\ &= \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{v}_k(u\cdot k)e^{i(2\pi/L)k\cdot x}\\ &= \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{v}_k\left(\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\hat{u}_j e^{i(2\pi/L)j\cdot x}\cdot k\right)e^{i(2\pi/L)k\cdot x}\\ &= \frac{2\pi i}{L}\sum_{j,k\in\mathbb{Z}^n\setminus\{0\}}(\hat{u}_j\cdot k)\,\hat{v}_k e^{i(2\pi/L)(k+j)\cdot x}\end{aligned}$$

Now we reindex with $m = k + j$ to get

$$(u\cdot\nabla)v = \frac{2\pi i}{L}\sum_{m\in\mathbb{Z}^n\setminus\{0\}}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}(\hat{u}_{m-k}\cdot k)\,\hat{v}_k e^{im\cdot x}$$

To get $B(u, v)$ we need to project $(u\cdot\nabla)v$ onto the divergence-free vector fields. On the torus, this means we need $\widehat{B(u,v)}_m\cdot m = 0$ for each $m\in\mathbb{Z}^n\setminus\{0\}$. Since $L^2$ can be orthogonally decomposed into gradients and divergence-free vector fields, we can calculate $B(u, v)$ in two ways. First, we may subtract off the projection of $(u\cdot\nabla)v$ onto the space of gradients. In this case we project $[\widehat{(u\cdot\nabla)}v]_m$ onto $m$ and then subtract. Thus we have

$$\widehat{B(u,v)}_m = \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\left[(\hat{u}_{m-k}\cdot k)\hat{v}_k - \frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m)}{|m|^2}m\right].$$

Alternatively, in 2D we can project $[\widehat{(u\cdot\nabla)}v]_m$ onto the direction of $m^\perp = (-m_2, m_1)$ as follows:

$$\widehat{B(u,v)}_m = \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m^\perp)}{|m^\perp|^2}m^\perp.$$

Since $|m^\perp| = |m|$ we may rewrite this as

$$\widehat{B(u,v)}_m = \frac{2\pi i}{L}\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m^\perp)}{|m|^2}m^\perp.$$

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