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Research Statement

Introduction

I love doing research in an area of mathematics where beautiful mathematical theory is grounded in concrete physical interpretations. My research falls under the broad umbrella of mathematical fluid dynamics. In particular, I study the equations that describe the evolution of the motion of fluids—a system of partial differential equations known as the incompressible Navier-Stokes equations (NSE). These equations are famous because of a remarkable contradiction in practice and theory: the Navier-Stokes equations are applied with success in areas ranging from blood flow to weather prediction, and yet not only is there no explicit solution for these equations in general, it is not even known whether the problem is well-posed! This significant gap between theory and practice is why the Clay Institute lists the resolution of the existence and uniqueness of physically realistic solutions as one of its seven Millennium Prizes. However, even if the millennium problem is solved tomorrow, there are still mountains of open problems of interest to mathematicians and scientists alike concerning the general behavior of the solutions that do exist. Indeed, in the case of 2 dimensions the existence/uniqueness problem has been solved, and yet little is known about the qualitative properties of the solutions, especially in connection with applications.

In particular, turbulent fluid flow (a phenomenon familiar to anyone who has seen a crashing waterfall or experienced a choppy airplane flight) is poorly understood both physically and mathematically. The existing empirical theory of turbulent flow, while being appropriately descriptive, remains disconnected from the equations that govern the flow, and thus is inadequate for making predictions. Mathematically, we would like to ground the physical theory in the Navier-Stokes equations themselves in a way that is both explanatory and predictive. We would like to develop rigorous conditions that give rise to or preclude turbulence and to understand statistical behavior of turbulent phenomena. The empirical theory points towards the importance of two properties for understanding turbulence: kinetic energy (a measure of total fluid motion) and enstrophy (a measure of the "swirliness" of the fluid). As a result, much ongoing work is dedicated to understanding the evolution of these quantities in actual solutions. It is in this area that my research lies.

In my research, which has been partially funded by the National Science Foundation, I consider the incompressible Navier-Stokes equations from a dynamical systems perspective, with an eye towards understanding turbulence. The central question of my dissertation considers whether (given the right initial data and forcing function) it is possible to have a solution of the Navier-Stokes equations that is non-stationary but where the energy and enstrophy nevertheless remain constant in time. Such solutions have been dubbed "ghost solutions" in the literature [4]. I find the question of whether ghost solutions exist interesting on its face, but there are several independent motivations for investigating them.

Motivation

Taking a dynamical systems approach to the NSE means thinking about solutions geometrically. We consider the set of all 2D or 3D vector fields (over whatever spatial domain we want to consider) with finite energy (L^2 norm) and finite enstrophy (L^2 norm of the curl). The solution over time of the NSE can be thought of as a path in this space of possible vector fields. In the 2D case all roads lead to Rome, and there exists a compact set known as the *global attractor* that attracts all paths quickly and uniformly [see figure 1]. (There is a similar concept in 3D though the story is more complicated). The global attractor captures the long-term dynamics of the system and so is an object of great interest.

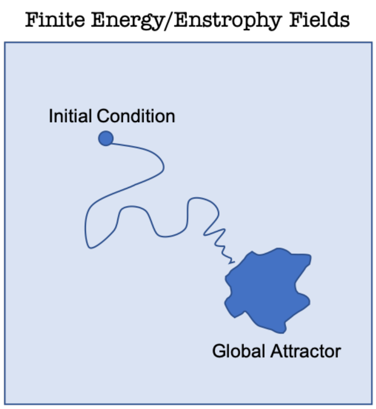

Figure 1

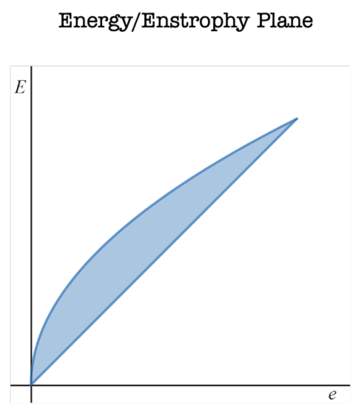

Figure 2

It was shown in [1] that, given the simplest meaningful spatial domain for the NSE, the energy and enstrophy of solutions in the global attractor lie somewhere within a closed parabolic region of the energy-enstrophy (e-E) plane [see figure 2]. A question that arises naturally from this result (as well as the empirical theory of turbulence) is just how well the dynamics of the global attractor can be captured by considering simply the dynamics in the energy/enstrophy plane. In particular, does a stationary point in the energy/enstrophy plane necessarily correspond to a stationary point in the space of possible solutions? Any non-stationary solution whose projection into the energy/enstrophy plane is stationary would be a ghost solution. Thus, the existence of ghost solutions would be an important check on the utility of simply considering the energy/enstrophy of a solution in order to understand its dynamics.

Another motivation for the study of ghost solutions comes from results in [3] that give a sufficient condition for generating a robust type of turbulence in 2D that has remained elusive in numerical simulations. It is unknown whether such turbulence is theoretically ruled out for some reason, or whether certain simplifications in the simulation process preclude its development. The theoretical results from [3] show that solutions where the ratio of the *average* energy to the *average* enstrophy stays small enough must necessarily generate the elusive turbulence. Ghost solutions would be nice candidates for such solutions since the ratio of their energy to enstrophy is constant.

Results and Future Work

Significant results concerning ghost solutions are minimal in the literature. It is known that if the norm of the force is small enough, then there are no ghost solutions in the global attractor [1]. It is also known that if the force has an extremely special Fourier representation then there are no ghost solutions in the global attractor [5]. In both of these situations we also know that there is no turbulence. More recent results [6, 7] focus on ghost solutions that satisfy an additional (somewhat ad hoc) condition that makes their analysis easier. These are referred to as *chained ghost solutions* in the literature. Their existence is ruled out under only a handful of extremely specific conditions. Unknown still are whether more generally there exist non-stationary solutions simply with constant energy profiles. Similarly, it is unknown whether a non-stationary solution may exhibit constant enstrophy regardless of the energy profile.

My own results in this area are as follows. I can construct a non-stationary solution in 3D such that the energy remains constant. I can also construct a non-stationary solution in 3D where the enstrophy remains constant. However, these constructions preclude keeping both properties constant, and they necessarily live outside of the attractor. These constructions come from studying a simpler system, known as the Stokes equations, where the constructions are more straightforward and a complete theory of ghost solutions can be established [2]. In a different line of attack, relying heavily on Fourier analysis, I can show that chained ghost solutions do not exist under any condition. My results on chained ghost solutions have implications for what solutions to the NSE can look like more generally. My current research is directed towards understanding these consequences and hopefully expanding their scope. In my future work I plan to focus study on the ratio of average energy to average enstrophy described in [3]. The hope is to come up with similar sufficient conditions for turbulence but in terms of the geometric structure of the forcing function.

References

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