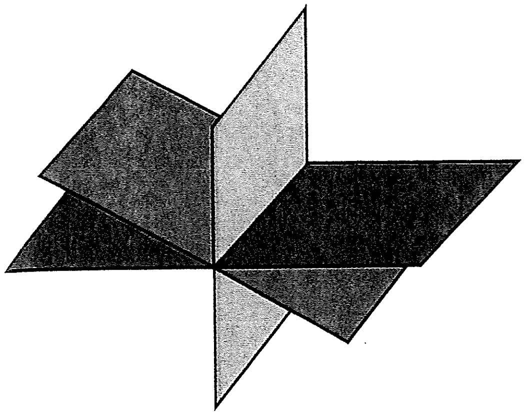
CHAPTER 1



Linear Equations

1.1 INTRODUCTION TO LINEAR SYSTEMS

Traditionally, algebra was the art of solving equations and systems of equations. The word *algebra* comes from the Arabic *al-jabr,* which means *restoration* (of broken parts).[1] The term was first used in a mathematical sense by Mohammed al-Khowarizmi (c. 780–850), who worked at the House of Wisdom, an academy established by Caliph al-Ma'mun in Baghdad. Linear algebra, then, is the art of solving systems of linear equations.

The need to solve systems of linear equations frequently arises in mathematics, statistics, physics, astronomy, engineering, computer science, and economics.

Solving systems of linear equations is not conceptually difficult. For small systems, ad hoc methods certainly suffice. Larger systems, however, require more systematic methods. The approach generally used today was beautifully explained 2,000 years ago in a Chinese text, the *Nine Chapters on the Mathematical Art* (Jiuzhang Suanshu, 九章算術).[2] Chapter 8 of that text, called *Method of Rectangular Arrays* (Fang Cheng, 方程), contains the following problem:

> The yield of one bundle of inferior rice, two bundles of medium grade rice, and three bundles of superior rice is 39 *dou* of grain.[3] The yield of one bundle of inferior rice, three bundles of medium grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium grade rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

In this problem the unknown quantities are the yields of one bundle of inferior, one bundle of medium grade, and one bundle of superior rice. Let us denote these quantities by x, y, and z, respectively. The problem can then be represented by the

[1] At one time, it was not unusual to see the sign *Algebrista y Sangrador* (bone setter and blood letter) at the entrance of a Spanish barber's shop.

[2] Shen Kangshen et al. (ed.), *The Nine Chapters on the Mathematical Art*, Companion and Commentary, Oxford University Press, 1999.

[3] The *dou* is a measure of volume, corresponding to about 2 liters at that time.

following system of linear equations:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

To solve for x, y, and z, we need to transform this system from the form

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \text{into the form} \quad \begin{vmatrix} x \quad \quad = \ldots \\ \quad y \quad = \ldots \\ \quad \quad z = \ldots \end{vmatrix}.$$

In other words, we need to eliminate the terms that are off the diagonal, those circled in the following equations, and make the coefficients of the variables along the diagonal equal to 1:

$$\begin{array}{ccccccc} x & + & \textcircled{2y} & + & \textcircled{3z} & = & 39 \\ \textcircled{x} & + & 3y & + & \textcircled{2z} & = & 34 \\ \textcircled{3x} & + & \textcircled{2y} & + & z & = & 26. \end{array}$$

We can accomplish these goals step by step, one variable at a time. In the past, you may have simplified systems of equations by adding equations to one another or subtracting them. In this system, we can eliminate the variable x from the second equation by subtracting the first equation from the second:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \xrightarrow{-\text{1st equation}} \quad \begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

To eliminate the variable x from the third equation, we subtract the first equation from the third equation three times. We multiply the first equation by 3 to get

$$3x + 6y + 9z = 117 \qquad (3 \times \text{1st equation})$$

and then subtract this result from the third equation:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \xrightarrow[-3 \times \text{1st equation}]{} \quad \begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{vmatrix}.$$

Similarly, we eliminate the variable y above and below the diagonal:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{vmatrix} \quad \xrightarrow[+4 \times \text{2nd equation}]{-2 \times \text{2nd equation}} \quad \begin{vmatrix} x \quad + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{vmatrix}.$$

Before we eliminate the variable z above the diagonal, we make the coefficient of z on the diagonal equal to 1, by dividing the last equation by -12:

$$\begin{vmatrix} x \quad + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{vmatrix} \quad \xrightarrow[\div(-12)]{} \quad \begin{vmatrix} x \quad + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{vmatrix}.$$

Finally, we eliminate the variable z above the diagonal:

$$\begin{vmatrix} x \quad + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{vmatrix} \quad \xrightarrow{\substack{-5 \times \text{third equation} \\ + \text{third equation}}} \quad \begin{vmatrix} x \quad \quad = 2.75 \\ y \quad = 4.25 \\ z = 9.25 \end{vmatrix}.$$

The yields of inferior, medium grade, and superior rice are 2.75, 4.25, and 9.25 *dou* per bundle, respectively.

By substituting these values, we can check that $x = 2.75$, $y = 4.25$, $z = 9.25$ is indeed the solution of the system:

$$\begin{aligned} 2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\ 2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\ 3 \times 2.75 + 2 \times 4.25 + \quad 9.25 &= 26. \end{aligned}$$

Happily, in linear algebra, you are almost always able to check your solutions. It will help you if you get into the habit of checking now.

Geometric Interpretation

How can we interpret this result geometrically? Each of the three equations of the system defines a plane in x–y–z-space. The solution set of the system consists of those points (x, y, z) that lie in all three planes (i.e., the intersection of the three planes). Algebraically speaking, the solution set consists of those ordered triples of numbers (x, y, z) that satisfy all three equations simultaneously. Our computations show that the system has only one solution, $(x, y, z) = (2.75, 4.25, 9.25)$. This means that the planes defined by the three equations intersect at the point $(x, y, z) = (2.75, 4.25, 9.25)$, as shown in Figure 1.

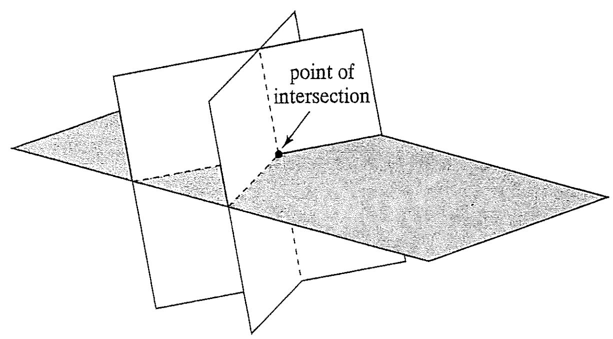


Figure 1 Three planes in space, intersecting at a point.

While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.

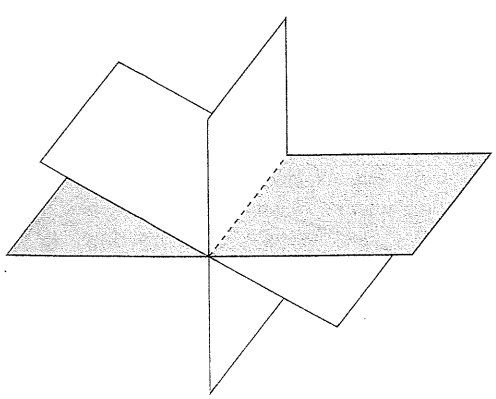


Figure 2(a) Three planes having a line in common.

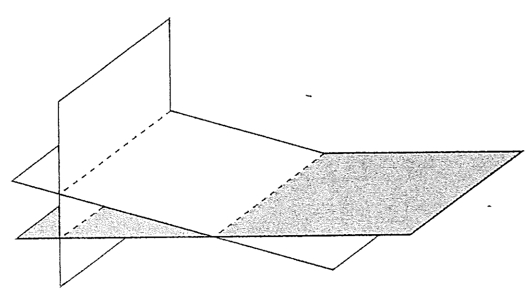


Figure 2(b) Three planes with no common intersection.

A System with Infinitely Many Solutions

Next, let's consider a system of linear equations that has infinitely many solutions:

$$\left| \begin{aligned} 2x + 4y + 6z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned} \right|.$$

We can solve this system using elimination as previously discussed. For simplicity, we label the equations with Roman numerals.

$$\left| \begin{aligned} 2x + 4y + 6z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned} \right| \quad \begin{matrix} \div 2 \\ \longrightarrow \\ \end{matrix} \quad \left| \begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned} \right| \quad \begin{matrix} \longrightarrow \\ -4\text{ (I)} \\ -7\text{ (I)} \end{matrix}$$

$$\left| \begin{aligned} x + 2y + 3z &= 0 \\ -3y - 6z &= 3 \\ -6y - 12z &= 6 \end{aligned} \right| \quad \begin{matrix} \longrightarrow \\ \div(-3) \\ \end{matrix} \quad \left| \begin{aligned} x + 2y + 3z &= 0 \\ y + 2z &= -1 \\ -6y - 12z &= 6 \end{aligned} \right| \quad \begin{matrix} -2\text{ (II)} \\ \longrightarrow \\ +6\text{ (II)} \end{matrix}$$

$$\left| \begin{aligned} x - z &= 2 \\ y + 2z &= -1 \\ 0 &= 0 \end{aligned} \right| \quad \longrightarrow \quad \left| \begin{aligned} x - z &= 2 \\ y + 2z &= -1 \end{aligned} \right|$$

After omitting the trivial equation $0 = 0$, we are left with only two equations with three unknowns. The solution set is the intersection of two nonparallel planes in space (i.e., a line). This system has infinitely many solutions.

The two foregoing equations can be written as follows:

$$\left| \begin{aligned} x &= z + 2 \\ y &= -2z - 1 \end{aligned} \right|.$$

We see that both x and y are determined by z. We can freely choose a value of z, an arbitrary real number; then the two preceding equations give us the values of x and y for this choice of z. For example,

- Choose $z = 1$. Then $x = z + 2 = 3$ and $y = -2z - 1 = -3$. The solution is $(x, y, z) = (3, -3, 1)$.
- Choose $z = 7$. Then $x = z + 2 = 9$ and $y = -2z - 1 = -15$. The solution is $(x, y, z) = (9, -15, 7)$.

More generally, if we choose $z = t$, an arbitrary real number, we get $x = t + 2$ and $y = -2t - 1$. Therefore, the general solution is

$$(x, y, z) = (t + 2, -2t - 1, t) = (2, -1, 0) + t(1, -2, 1).$$

This equation represents a line in space, as shown in Figure 3.

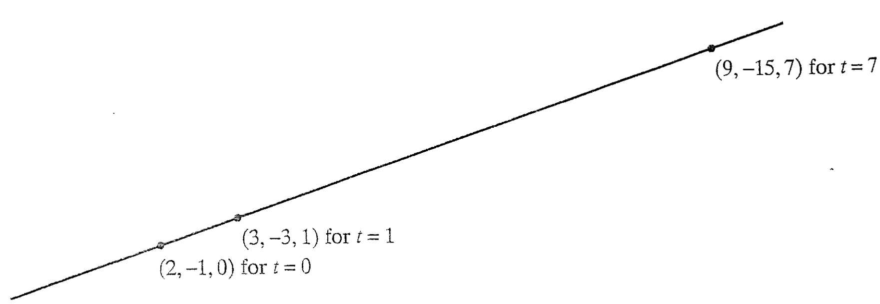


Figure 3 The line $(x, y, z) = (t + 2, -2t - 1, t)$.

A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\left|\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 0 \end{aligned}\right| \longrightarrow \left|\begin{aligned} x \qquad - z &= 2 \\ y + 2z &= -1 \\ 0 &= -6 \end{aligned}\right|.$$

Whatever values we choose for x, y, and z, the equation $0 = -6$ cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

EXERCISES 1.1

GOAL Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\left|\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 1 \end{aligned}\right|$

2. $\left|\begin{aligned} 4x + 3y &= 2 \\ 7x + 5y &= 3 \end{aligned}\right|$

3. $\left|\begin{aligned} 2x + 4y &= 3 \\ 3x + 6y &= 2 \end{aligned}\right|$

4. $\left|\begin{aligned} 2x + 4y &= 2 \\ 3x + 6y &= 3 \end{aligned}\right|$

5. $\left|\begin{aligned} 2x + 3y &= 0 \\ 4x + 5y &= 0 \end{aligned}\right|$

6. $\left|\begin{aligned} x + 2y + 3z &= 8 \\ x + 3y + 3z &= 10 \\ x + 2y + 4z &= 9 \end{aligned}\right|$

7. $\left|\begin{aligned} x + 2y + 3z &= 1 \\ x + 3y + 4z &= 3 \\ x + 4y + 5z &= 4 \end{aligned}\right|$

8. $\left|\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 10z &= 0 \end{aligned}\right|$

9. $\left|\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 2y + z &= 1 \\ 7x + 2y - 3z &= 1 \end{aligned}\right|$

10. $\left|\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 7z &= 2 \\ 3x + 7y + 11z &= 8 \end{aligned}\right|$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\left|\begin{aligned} x - 2y &= 2 \\ 3x + 5y &= 17 \end{aligned}\right|$

12. $\left|\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}\right|$

13. $\left|\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 8 \end{aligned}\right|$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\left|\begin{aligned} x + 4y + z &= 0 \\ 4x + 13y + 7z &= 0 \\ 7x + 22y + 13z &= 1 \end{aligned}\right|$

15. $\left|\begin{aligned} x + y - z &= 0 \\ 4x - y + 5z &= 0 \\ 6x + y + 4z &= 0 \end{aligned}\right|$

16. $\left|\begin{aligned} x + 4y + z &= 0 \\ 4x + 13y + 7z &= 0 \\ 7x + 22y + 13z &= 0 \end{aligned}\right|$

17. Find all solutions of the linear system

$$\left|\begin{aligned} x + 2y &= a \\ 3x + 5y &= b \end{aligned}\right|,$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\left|\begin{aligned} x + 2y + 3z &= a \\ x + 3y + 8z &= b \\ x + 2y + 2z &= c \end{aligned}\right|,$$

where a, b, and c are arbitrary constants.

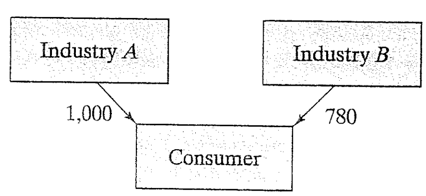
19. Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2, are given by

$$\begin{aligned} D_1 &= 70 - 2P_1 + P_2, & S_1 &= -14 + 3P_1, \\ D_2 &= 105 + P_1 - P_2, & S_2 &= -7 + 2P_2. \end{aligned}$$

a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?

b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.



What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?