the names of the various conditions, including the following two samples.

DEFINITIONS 4.2.5. A topological space is *regular* (*normal*) if given any closed subset V and x in $T-V$ (closed subset $V'\subset T-V$), there exist disjoint open subsets U, U' of T such that $V\subset U$, $x\in U'$ ($V'\subset U'$).

Exercises 4.3

1. Show that the Zariski topology of Example 3.1.7 is not Hausdorff.
2. Prove Proposition 4.2.4.
3. Prove that if $f: T_1\to T_2$ is a continuous map of a topological space T_1 to a Hausdorff space T_2, then the graph G_f is a closed subset of $T_1\times T_2$.
4. (a) Prove that if x is a point in a Hausdorff space T, then the intersection of all open subsets of T containing x is $\{x\}$.
   (b) Give an example to show that the conclusion of (a) does not imply that T is Hausdorff. (Hint: Exercise 1.)

5*. Let M be a metric space with metric d, and for any non-empty subset K and point x in M, let $d(x, K)=\inf_{y\in K} d(x, y)$. Prove that
   (a) if K is closed in M then $d(x, K)=0$ iff $x\in K$,
   (b) $d(x, K)\leq d(x, y)+d(y, K)$ for any x, y in M,
   (c) the map $f_K: M\to \mathbf{R}$ given by $f_K(x)=d(x, K)$ is continuous.
   Suppose that H, K are disjoint closed sets in M. Let $g: M\to\mathbf{R}$ be given by $g=f_K-f_H$. Prove that $g^{-1}(0, \infty)$ and $g^{-1}(-\infty, 0)$ are disjoint open sets containing H, K respectively. (This shows that any metric space is normal.)

6. Let f, $g: T_1\to T_2$ be continuous maps of a topological space T_1 to a Hausdorff space T_2. Show that $W=\{x\in T_1: f(x)=g(x)\}$ is closed in T_1. Deduce that if $f: T\to T$ is a continuous map of a Hausdorff space T, then the 'fixed-point set' $\{x\in T: f(x)=x\}$ is closed in T.

5

Compact spaces

5.1 Motivation

THE subject matter of this chapter is probably the most important single topic in this book. There are several ways of framing the definition of compactness. The definition in this chapter is appropriate for topological spaces. Another important definition which works well in metric spaces will be studied in Chapter 7 and related to the present definition.

Recall from the introduction that we are aiming to prove some basic results about continuous functions in a general setting, and that the following is an example of the kind of result we wish to generalize.

PROPOSITION 5.1.1. *A continuous function $f:[a, b]\to\mathbf{R}$ is bounded on $[a, b]$.*

We shall begin with a slow build-up towards one way of proving this, and generalizations of it. Let us set out by supposing that the function $f: A\to\mathbf{R}$ is defined on some general subset A of $\mathbf{R}$. We may ask, 'Is f bounded on A?', that is, 'Does there exist a fixed real number K such that $|f(x)|\leq K$ for all x in A?' We shall proceed from the known to the unknown in examining this question.

STEP 1. If A is a finite set, say $A=\{a_1, a_2, \ldots, a_r\}$, then the answer is 'Yes', and we may take

$$K=\text{Max}\{|f(a_1)|, |f(a_2)|, \ldots, |f(a_r)|\}.$$

STEP 2. More generally, if A is a finite union of subsets, say $A=\bigcup_{i=1}^{r} A_i$, and if we already know that f is bounded on each A_i separately, say $|f(x)|\leq K_i$ for all x in A_i, then again f is bounded on A: we set $K=\text{Max}\{K_1, K_2, \ldots, K_r\}$, and then for any x in A, $x\in A_i$ for some i, so $|f(x)|\leq K_i\leq K$.

EXAMPLE 5.1.2. We now look at an example of a function which is not bounded. Let $A=(0, 1)$ and put $f(x)=1/x$ for all x in $(0, 1)$. Given any real number $K>0$, we can find a real number x such that $0<x<1/K$, (and $x<1$) and then $f(x)=1/x>K$. So no K is large enough to bound $f(x)$ on all of $(0, 1)$. This illustrates that the answer to our question can be negative even when f is continuous.

STEP 3. However, we get *something* when f is continuous. For suppose that $f: A\to \mathbf{R}$ is continuous, and for any a in A let us apply the $\varepsilon-\delta$ definition of continuity with $\varepsilon=1$, say. Thus there exists some $\delta>0$ such that $|f(x)-f(a)|<1$ for all x in A satisfying $|x-a|<\delta$. So $|f(x)|<1+|f(a)|$ for all x in $B_\delta(a)$. The δ here in general depends on a and on f as well as on the chosen value 1 of ε. Let us write it in the meantime as $\delta(a)$. So for continuous f, given any a in A, there is a single bound $K_a=1+|f(a)|$ for $|f(x)|$ which serves for all x in some neighbourhood $B_{\delta(a)}(a)$ of a.

STEP 4. Now recall that the original question is whether there is a single bound for $|f(x)|$ which serves for all x in A. We cannot in general answer this affirmatively by taking the maximum of the K_a in Step 3, because in general there are infinitely many K_a involved (one for each a in A) and the set of these K_a may not be bounded, far less have a maximum. However, suppose for a moment that A is the union of some *finite* number of the $B_{\delta(a)}(a)$ occurring in Step 3. Then, since f is bounded on each $B_{\delta(a)}(a)$, it follows by Step 2 that f is bounded on A.

Let us examine more closely the assumption which allowed us to reach this conclusion. Originally we just had $A=\bigcup_{a\in A} B_{\delta(a)}(a)$, and we then assumed that finitely many of these neighbourhoods were sufficient to give A as their union. Now the neighbourhoods $B_{\delta(a)}(a)$ depended on f as well as on a, so if we want to set down a condition *on* A which will enable us to prove by the above argument that *any* continuous real-valued function on A is bounded, we had better assume something like the property defined as follows.

PROVISIONAL DEFINITION 5.1.3. A subset A of $\mathbf{R}$ is *compact* if whenever A is the union of a collection of open balls, it is a union of finitely many of the balls in that collection.

Before stating the definition more precisely, let us consider what compactness enables us to do. The conclusion of Step 3, that any continuous real-valued function is bounded on some neighbourhood of each point in its domain, may be called a local statement, because it makes an assertion only about some neighbourhood of each point. The statement that f is bounded on A, however, may be called a global statement because it describes a property of f on the whole domain A. Compactness of A allows us to pass from the local to the global.

As it stands, Definition 5.1.3 makes sense for any metric space. As usual, replacing open balls by open sets generalizes it to topological spaces.

5.2 Definition of compactness

The definition of compactness may conveniently be expressed in the language of covers.

DEFINITIONS 5.2.1. A *cover* for a set A is a collection $\mathcal{U}$ of sets such that $A\subset \bigcup_{U\in\mathcal{U}} U$. A *subcover* of a given cover $\mathcal{U}$ for A is a subcollection $\mathcal{V}\subset\mathcal{U}$ which still forms a cover for A. A cover $\mathcal{U}$ is *finite* if it is finite, that is, if there are only finitely many sets in the collection $\mathcal{U}$. If A is a subspace of a topological space T, and if $\mathcal{U}$ is a cover for A in which each set U is open in T, then we say that $\mathcal{U}$ is an *open* cover for A (in particular A can equal T in this definition).

DEFINITION 5.2.2. A topological space T is *compact* if every open cover of T has a finite subcover.

It is important to notice exactly what this definition says: given any open cover $\mathcal{U}$ of T, there exists a finite number of the open sets in $\mathcal{U}$ which are enough to cover T. This is very different, for example, from saying that there exists a finite open cover of T. Indeed, the latter assertion is trivially true for any topological space T, for we may take the singleton collection $\{T\}$ as a finite open cover.

To emphasize the definition further, let us see that $(0, 1)$ is not compact. Consider the collection of open intervals $(1/n, 1)$ for all integers $n\geqslant 2$. Each of these is open in $(0, 1)$, and $(0, 1)=\bigcup_{n=2}^{\infty}(1/n, 1)$. (Note that given any x in $(0, 1)$, there exists an integer n such that $n>1/x$, and then $x\in(1/n, 1)$.) But no finite subcollection of this particular collection will suffice to cover $(0, 1)$, since the